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Solution of Newton's gravitational field equation of a static homogeneous Prolate spheroidal massive body

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Abstract: ***It is well known that Newton's dynamical gravitational field equation and Einstein's geometrical gravitational field equation were derived based upon the Euclidean geometry. In this paper, we used a golden metric tensor for all gravitational fields in nature to develop Riemannian Laplacian field equation which is assumed to be more general than the Euclidean geometry in order to derive Riemann's dynamical gravitational field equation for static homogeneous Prolate spheroidal massive bodies due correction terms of all order*** c^{-2} ***that is not found in Newton's***

gravitational field equation and solution of Newton's gravitational field equation as an exact expression for a static homogeneous prolate spheroidal massive body as an extension of the gravitational fields of spherical body are been derived for investigations and applications. The expressions for the gravitational fields of spherical bodies are well known.

Keywords: **Golden metric tensor, Riemannian Laplacian Equation, Spherical polar coordinates, Oblate spheroidal coordinates.**

I. INTRODUCTION

In the present paper some theoretical background of gravitational field of spheroidal bodies is stated. Also in this paper the gravitational scalar potential for both the interior and exterior regions are evaluated by means of Legendre differential equations. Then the gravitational intensities of prolate spheroidal bodies for both the regions are determined.

Prior to 1950, theoretical study of gravitation was restricted almost exclusively to the fields of massive bodies of perfect spherical geometry. For example, in Newton's theory of universal gravitation (T.U.G) the motion of particles (such as projectile, satellites and pendulli) in earth's atmosphere is treated under the assumption that the earth is a perfect sphere. Similarly, in the solar system the motion of bodies (such as planets comets and asteroids) is treated under the assumption that the Sun is a perfect sphere.

Also in Einstein's theory of gravitation, called General Relativity Theory (G.R.T.) the motion of bodies (such as planets) and particles (such as photons) is treated under the assumption that the Sun is a perfect sphere (Schwarzschild space time). But it is well known that the only reason for these restrictions is mathematical convenience and simplicity. The real fact of nature is that all rotating planets stars and galaxies in the universe are spheroidal. And it is obvious that their spheroidal geometry will have corresponding consequences and effects in the motion of all particles in their gravitational fields. These effects will exist in both Newton's T.U.G and in Einstein's G.R.T. Consequently the way is prepared for the study of motion of all particles in the gravitational fields of spheroidal bodies.

As an example it is now well known that satellite orbits around earth are not governed by just the simple universe distance squared gravitational field of perfect spherical geometry. They are also governed by second harmonics (pole of order 3) as well as fourth, harmonics (pole of order 5) of gravitational scalar potential due to imperfect spherical geometry. Now for comparison with these approximations the exact analytical gravitational scalar potential for a perfect oblate spheroidal body is derived (1, 2, 3, 4).

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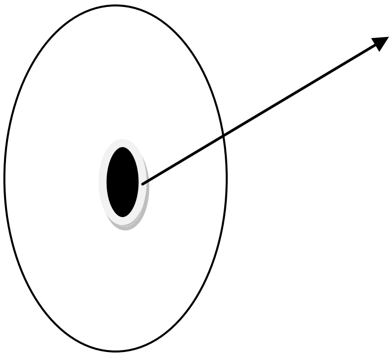
II. THEORETICAL ANALYSIS

Consider a static homogeneous Prolate spheroidal massive body of rest mass M and radius η_0. The Riemannian dynamical gravitational field equation is given explicitly by [5, 6, 7, 8, 9, 10, 11]



Density ρ_0

Radius η_0

Mass M

$$\nabla_R^2 f(\eta, \xi, \phi, x^0) = 4\pi G \rho_0(\eta, \xi, \phi, x^0) \qquad (1.0)$$

Where ∇_R^2 is Riemannian Laplacian

Given

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left\{ \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right\} \qquad (1.1)$$

and g_{uv} is the metric tensor (either Schwarzschild or great metric tensor or golden metric tensor). Here we use golden metric tensor of prolate spheroidal coordinates. In spherical polar coordinates, Golden metric tensor is given by:

$$g_{11} = \left\{ 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right\}^{-1} \qquad (1.2)$$

$$g_{22} = r^2 \left\{ 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right\}^{-1} \qquad (1.3)$$

$$g_{33} = r^2 \sin^2\theta \left\{ 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right\}^{-1} \qquad (1.4)$$

$$g_{00} = -\left\{ 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right\} \qquad (1.5)$$

$$g_{uv} = 0, \text{otherwise} \qquad (1.6)$$

To express Golden metric tensor in prolate spheroidal coordinates, we need to transform as we shall see later. $(r, \theta, \phi, x^0) \rightarrow (\eta, \xi, \phi, x^0)$

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Note that:

$$g_{11} = h_1^2\left(1+\frac{2}{c^2}f\right)^{-1} \quad (1.7)$$

$$g_{22} = h_2^2\left(1+\frac{2}{c^2}f\right)^{-1} \quad (1.8)$$

$$g_{33} = h_3^2\left(1+\frac{2}{c^2}f\right)^{-1} \quad (1.9)$$

$$g_{00} = -\left(1+\frac{2}{c^2}f\right) \quad (1.10)$$

$$g_{uv} = 0, \text{otherwise} \quad (1.11)$$

Where h_1, h_2, h_3 are the scale factors in prolate spheroidal coordinates to be evaluated.

Now consider Riemannian Laplacian

$$\nabla_R^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{1\beta}\frac{\partial}{\partial x^\beta}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{2\beta}\frac{\partial}{\partial x^\beta}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{3\beta}\frac{\partial}{\partial x^\beta}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{0\beta}\frac{\partial}{\partial x^\beta}\right\} \quad (1.12)$$

$$\nabla_R^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{11}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{12}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{13}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{10}\frac{\partial}{\partial x^0}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{21}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{22}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{23}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{20}\frac{\partial}{\partial x^0}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{31}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{32}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{33}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{30}\frac{\partial}{\partial x^0}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{01}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{02}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{03}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{00}\frac{\partial}{\partial x^0}\right\}$$

$$= 4\pi G\rho_0(r, \theta, \phi, x^0) \quad (1.13)$$

The above Riemannian field equation reduced to:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{11}\frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{22}\frac{\partial}{\partial x^2}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{33}\frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{00}\frac{\partial}{\partial x^0}\right\}$$

$$= 4\pi G\rho_0(r, \theta, \phi, x^0) \quad (1.14)$$

By definition g is the determinant of the Golden metric tensor

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$$g = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{00} \end{pmatrix} \quad (1.15)$$

$$g = g_{11} \cdot g_{22} \cdot g_{33} \cdot g_{00} \quad (1.16)$$

$$= -h_1^2\ h_2^2\ h_3^2 \left(1 + \frac{2}{c^2} f\right)^{-3} \left(1 + \frac{2}{c^2} f\right)^{1} = -h_1^2\ h_2^2\ h_3^2 \left(1 + \frac{2}{c^2} f\right)^{-2}$$

$$\sqrt{g} = h_1\ h_2\ h_3 \left(1 + \frac{2}{c^2} f\right)^{-1} \quad (1.17)$$

For prolate spheroidal coordinate

$$g^{11} = \frac{1}{g_{11}} = \frac{1}{h_1^2 \left(1 + \frac{2}{c^2} f\right)^{-1}} = \frac{\left(1 + \frac{2}{c^2} f\right)}{h_1^2} \quad (1.18)$$

$$g^{22} = \frac{1}{g_{22}} = \frac{1}{h_2^2 \left(1 + \frac{2}{c^2} f\right)^{-1}} = \frac{\left(1 + \frac{2}{c^2} f\right)}{h_2^2} \quad (1.19)$$

$$g^{33} = \frac{1}{g_{33}} = \frac{1}{h_3^2 \left(1 + \frac{2}{c^2} f\right)^{-1}} = \frac{\left(1 + \frac{2}{c^2} f\right)}{h_3^2} \quad (1.20)$$

$$g^{00} = \frac{1}{g_{00}} = -\frac{1}{\left(1 + \frac{2}{c^2} f\right)} = -\left(1 + \frac{2}{c^2} f\right) \quad (1.21)$$

$$g^{uv} = 0, \quad \text{otherwise} \quad (1.22)$$

Note also the scale factors for prolate spheroidal coordinates are:

$$h_1 = a \left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^{1/2} \quad (1.23)$$

$$h_2 = a \left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^{1/2} \quad (1.24)$$

$$h_3 = a[(\xi^2 - 1)(1 - \eta^2)]^{1/2} \quad (1.25)$$

By making use of equation (1.7) above and substituting h_1, h_2 and h_3 . We get:

$$\sqrt{g} = a \left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^{1/2} \cdot a \left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^{1/2} \cdot a[(\xi^2 - 1)(1 - \eta^2)]^{1/2} \times \left(1 + \frac{2}{c^2} f\right)^{-1}$$

$$= [a^6 (\xi^2 - \eta^2)^2]^{1/2} \left(1 + \frac{2}{c^2} f\right)^{-1} \quad (1.26)$$

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Volume 6, Issue 1, January 2017

$$\sqrt{g} = a^3(\xi^2 - \eta^2)\left(1 + \frac{2}{c^2}f\right)^{-1} \tag{1.27}$$

Putting $\sqrt{g}$, g^{11}, g^{22}, g^{33} and g^{00} into the field equation we get

$$\frac{1}{a^3(\xi^2 - \eta^2)\left(1 + \frac{2}{c^2}f\right)^{-1}}\frac{\partial}{\partial\eta}\left\{a^3(\xi^2 - \eta^2)\left(1 + \frac{2}{c^2}f\right)^{-1}\frac{\left(1 + \frac{2}{c^2}f\right)}{h_1^2}\frac{\partial}{\partial\eta}f\right\}$$

$$+\frac{1}{a^3(\xi^2 - \eta^2)\left(1 + \frac{2}{c^2}f\right)^{-1}}\frac{\partial}{\partial\xi}\left\{a^3(\xi^2 - \eta^2)\left(1 + \frac{2}{c^2}f\right)^{-1}\frac{\left(1 + \frac{2}{c^2}f\right)}{h_2^2}\frac{\partial}{\partial\xi}f\right\}$$

$$+\frac{1}{a^3(\xi^2 - \eta^2)(1 + \frac{2}{c^2}f)^{-1}}\frac{\partial}{\partial\phi}\left\{a^3(\xi^2 - \eta^2)\left(1 + \frac{2}{c^2}f\right)^{-1}\frac{\left(1 + \frac{2}{c^2}f\right)}{h_3^2}\frac{\partial}{\partial\phi}f\right\}$$

$$-\frac{1}{a^3(\xi^2 - \eta^2)(1 + \frac{2}{c^2}f)^{-1}}\frac{\partial}{\partial x^0}\left\{\left(1 + \frac{2}{c^2}f\right)^{-1}\left(1 + \frac{2}{c^2}f\right)\frac{\partial}{\partial x^0}f\right\}$$

$$= 4\pi G\rho(\eta, \xi, \phi, x^0) \tag{1.28}$$

Note that

$$x^0 = ct \tag{1.29}$$

Static homogeneous prolate spheroidal distribution of massive body
Note: (1) No Time variation (static)

(2) No azimuthal angle ϕ (symmetry)

Equation (1.28) reduced to

$$\left(1 + \frac{2}{c^2}f\right)\left[\frac{\partial}{\partial\eta}\left\{(1 - \eta^2)\frac{\partial}{\partial\eta}f\right\} + \frac{\partial}{\partial\xi}\left\{(\xi^2 - 1)\frac{\partial}{\partial\xi}f\right\}\right] = 4\pi G\rho_0(\eta, \xi,)\, a^2(\xi^2 - \eta^2) \tag{1.31}$$

Expand: equation (1.31)

$$\frac{\partial}{\partial\eta}\left\{(1 - \eta^2)\frac{\partial}{\partial\eta}f\right\} + \frac{\partial}{\partial\xi}\left\{(\xi^2 - 1)\frac{\partial}{\partial\xi}f\right\} + \frac{2}{c^2}f\left[\frac{\partial}{\partial\eta}(1 - \eta^2)\frac{\partial}{\partial\eta}f\right] + \frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\{\xi^2 - 1\}\frac{\partial}{\partial\xi}f\right]$$

$$= a^2(\xi^2 - \eta^2)\cdot 4\pi G\rho_0(\eta, \xi,) \tag{1.32}$$

$$\frac{\partial}{\partial\eta}\left\{(1 - \eta^2)\frac{\partial}{\partial\eta}f\right\} + \frac{\partial}{\partial\xi}\left\{(\xi^2 - 1)\frac{\partial}{\partial\xi}f\right\} + \frac{2}{c^2}f\left[\frac{\partial}{\partial\eta}(1 - \eta^2)\frac{\partial}{\partial\eta}f\right] + \frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\{\xi^2 - 1\}\frac{\partial}{\partial\xi}f\right] = 0 \tag{1.33}$$

ISSN: 2319-5967
ISO 9001:2008 Certified
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Volume 6, Issue 1, January 2017

Note that:

- The first two terms of equation (1.32) are Newton's gravitational field equations interior to the gravitational scalar potential f due to a distribution of mass density $\rho_0(\eta,\xi,)$ and that of equation (1.33) are Newton's gravitational field equations exterior with no distribution of mass density.
- While the last two terms are the added terms or our contribution due to Riemann and is referred to as Riemann's gravitational field equation interior and exterior to a static homogeneous prolate spheroid.

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f(\eta,\xi,)\right\}+\frac{\partial}{\partial\xi}\left\{(\xi^2-1)\frac{\partial}{\partial\xi}f(\eta,\xi,)\right\}=4\pi G\rho_0 a^2(\xi^2-\eta^2) \quad (1.34)$$

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f(\eta,\xi,)\right\}+\frac{\partial}{\partial\xi}\left\{(\xi^2-1)\frac{\partial}{\partial\xi}f(\eta,\xi,)\right\}=0 \quad (1.35)$$

Gravitational fields of prolate spheroidal bodies

$$x=a[(1-\eta^2)(\xi^2-1)]^{\frac{1}{2}}\cos\phi \quad (1.36)$$

$$y=a[(1-\eta^2)(\xi^2-1)]^{\frac{1}{2}}\sin\phi \quad (1.37)$$

$$z=a\eta\xi \quad (1.38)$$

where a is a constant and $\{-1\le\eta\le 1;\ 0\le\xi<\infty;\ 0\le\phi\le 2\pi\}$. Since the body is homogeneous, the density of active mass ρ is given by

$$\rho(r)=\begin{cases}\rho_0 & \xi\le\xi_0\\ 0 & \xi>\xi_0\end{cases} \quad (1.39)$$

Where ρ_0 is the constant density of rest mass and the gravitational scalar potential of the body $f(\eta,\xi,\phi)$ is static and hence satisfied the field equation

$$\nabla^2 f(\eta,\xi,\phi)=4\pi G\rho(\eta,\xi,\phi) \quad (1.40)$$

The interior and exterior scalar potentials are given by

$$\frac{1}{a^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}+\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2}{\partial\phi^2}\right\}=4\pi G\rho_0(\eta,\xi,\phi) \quad (1.41)$$

and

$$\frac{1}{a^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}+\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2}{\partial\phi^2}\right\}=0 \quad (1.42)$$

A solution of variable separable, complementary independent of the azimuthal angle ϕ is given by

$$f_c(\eta,\xi)=\Omega(\eta)T(\xi) \quad (1.43)$$

Hence,

$$\left\{\begin{matrix}\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}f(\eta,\xi)\\ +\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}f(\eta,\xi)\end{matrix}\right\}=\left\{\begin{matrix}\frac{1}{\Omega(\eta)}\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}\Omega(\eta)\\ +\frac{1}{T(\xi)}\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}T(\xi)\end{matrix}\right\} \quad (1.44)$$

Rearranging and introducing a separation constant to the Laplace equation, we have

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International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 6, Issue 1, January 2017

$$\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta}\Omega(\eta) + \lambda\Omega(\eta) = 0 \qquad (1.45)$$

and

$$\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi}T(\xi) - \lambda T(\xi) = 0 \qquad (1.46)$$

where λ is the separation constants. Consequently, for the choice

$$\lambda = l(l+1);\ l = 0,1,2\ldots \qquad (1.47)$$

The solution of equations (1.45) and (1.46) of the Legendre's differential equation are

$$\Omega(\eta) = \begin{Bmatrix} P_l(\eta) \\ Q_l(\eta) \end{Bmatrix} \qquad (1.48)$$

and

$$T(\xi) = \begin{Bmatrix} P_l(\xi) \\ Q_l(\xi) \end{Bmatrix} \qquad (1.49)$$

Consequently

$$f(\eta,\xi) = \left\{\frac{2}{3}\pi\rho_0 a^2(\eta^2-\xi^2) + \sum_{l=0}^{\infty}[A_l P_l(\xi) + B_l Q_l(\xi)][C_l P_l(\eta) + D_l Q_l(\eta)]\right\} \qquad (1.50)$$

where A_l, B_l, C_l, D_l are constants and P_l, Q_l are the Legendre's functions of order l. Similarly the exterior homogeneous part has a solution given by

$$f(\eta,\xi) = \sum_{l=0}^{\infty}[A_l P_l(\xi) + B_l Q_l(\xi)][C_l P_l(\eta) + D_l Q_l(\eta)] \qquad (1.51)$$

Now since the interior and exterior regions both contain the coordinate $\eta = 0$ which is a singularity of Q_l

We choose

$$D_l = 0;\ l = 0,1,2,\ldots \qquad (1.52)$$

Also since Q_l is not defined at the centre of the body $\xi = 0$ in the interior region

$$B_l = 0;\ l = 0,1,2,\ldots \qquad (1.53)$$

Also, since P_l is not defined for $\xi \to \infty$ in the exterior region

$$A_l = 0;\ l = 0,1,2,\ldots \qquad (1.54)$$

Next the condition of the continuity of the potentials and their normal derivatives at $\xi = \xi_0$ (boundary of the spheroid), it follows that

$$B_0 = \frac{4\pi G\rho_0 a^2 \xi_0^2}{3\left[\frac{d}{d\xi}Q_0(\xi)\right]} \qquad (1.55)$$

and

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Volume 6, Issue 1, January 2017

$$A_0 = -\frac{2}{3}\pi\rho_0 a^2\left(\xi_0^2 + \frac{1}{3}\right) + \frac{4\pi G\rho_0 a^2\xi_0^2}{3\left[\frac{d}{d\xi}Q_0(\xi)\right]} \tag{1.56}$$

and $B_1 = A_1 = 0$

$$B_2 = \frac{4\pi G\rho_0 a^2\left[\frac{d}{d\xi}P_2(\xi)\right]}{9\left\{Q_2(\xi)\left[\frac{d}{d\xi}P_2(\xi)\right]_{\xi=\xi_0} - P_2(\xi)\left[\frac{d}{d\xi}Q_2(\xi)\right]_{\xi=\xi_0}\right\}} \tag{1.57}$$

and

$$A_2 = \frac{4\pi G\rho_0 a^2\left[\frac{d}{d\xi}P_2(\xi)\right]}{9\left\{Q_2(\xi)\left[\frac{d}{d\xi}P_2(\xi)\right]_{\xi=\xi_0} - P_2(\xi)\left[\frac{d}{d\xi}Q_2(\xi)\right]_{\xi=\xi_0}\right\}} \tag{1.58}$$

$B_l = A_l = 0$ for l=3,4,5,…

It follows that from (1.50) and (1.51) and (1.55) – (1.56), the potential are given by

$$f_c(\eta,\xi) = \frac{2\pi\rho_0 a^2}{3}\left\{\begin{array}{c} \xi^2 + \eta^2 + \left(\xi_0^2 + \frac{1}{3}\right) + \dfrac{2\xi_0 Q_0(\xi)P_0(\xi)P_0(\eta)}{\left[\frac{d}{d\xi}Q_0(\xi)\right]_{\xi=\xi_0}} \\ + \dfrac{2\left[\frac{d}{d\xi}Q_2(\xi)\right]_{\xi=\xi_0} P_2(\xi)P_2(\eta)}{3\left\{Q_2(\xi)\left[\frac{d}{d\xi}P_2(\xi)\right]_{\xi=\xi_0} - P_2(\xi)\left[\frac{d}{d\xi}Q_2(\xi)\right]_{\xi=\xi_0}\right\}} \end{array}\right\} \tag{1.59}$$

and

$$f(\eta,\xi) = \frac{4\pi\rho_0 a^2}{3}\left\{\begin{array}{c} \dfrac{\xi_0^2 Q_0(\xi)P_0(\xi)P_0(\eta)}{\left[\frac{d}{d\xi}Q_0(\xi)\right]} \\ + \dfrac{\left[\frac{d}{d\xi}P_2(\xi)\right]_{\xi=\xi_0} Q_2(\xi)P_2(\eta)}{3\left\{Q_2(\xi)\left[\frac{d}{d\xi}Q_0(\xi)\right] - P_2(\xi)\left[\frac{d}{d\xi}Q_2(\xi)\right]_{\xi=\xi_0}\right\}} \end{array}\right\} \tag{1.60}$$

The gravitational scalar potential (1.59) and (1.60) can be expressed in terms of the rest mass M_0 of the prolate spheroidal body is given by

$$M_0 = \frac{4}{9}a^3\rho_0\pi\xi_0\left(\xi_0^2 + 1\right) \tag{1.61}$$

It may be noted that results from mathematical tables gives

$$Q_0(t) = \frac{1}{2}\ln\left(\frac{t+1}{t-1}\right) \tag{1.62}$$

ISSN: 2319-5967
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International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 6, Issue 1, January 2017

$$Q_2(t) = \frac{1}{4}(3t^2 - 1)\ln\left(\frac{t+1}{t-1}\right) + \frac{3}{2}t \tag{1.63}$$

Consequently

$$Q_0(-i\xi) = i\left[\xi^{-1} + \frac{1}{3}\xi^{-3} + \frac{1}{5}\xi^{-5} + \cdots\right] \tag{1.64}$$

and

$$Q_2(-i\xi) = \frac{1}{2}\left[2\xi^{-1} + \left(\frac{1}{3} + \frac{1}{5}\right)\xi^{-3} + \left(\frac{1}{5} + \frac{1}{7}\right)\xi^{-5} + \cdots\right] \tag{1.65}$$

III. SUMMARY AND CONCLUSION

In this paper we showed how to formulate the Riemannian Gravitational Field Equation for static homogeneous prolate spheroidal massive bodies using the Riemannian Laplacian operator and the golden metric tensor. Also, in this research paper Newton's gravitational scalar potential for a homogeneous prolate spheroidal body was formulated and solved with exact and complete results given by (1.58) and (1.59). Consequently, these potentials are now available for application in physics.

A profound philosophical inference is suggested by the results obtained in the thesis that Newton's gravitational field equations for a spheroidal body are linear and separable, and hence solvable in terms of the well known special functions of mathematical physics, the Legendre, function. This fact suggests that Einstein's geometrical theory of gravitation, GRT, whose gravitational field equation are non linear may not be the most natural generalization of Newton's dynamical theory of gravitation.

It is now known that most bodies in the universe are spheroidal in nature. As an example, it is now well known that satellite orbits around the earth are not governed by just the simple inverse distance squared gravitational fields of perfect spherical geometry. They are also governed by second harmonics (pole of order 3), as well as fourth harmonics (pole of order 5) of gravitational scalar potential due to imperfect spherical geometry.

In the first place this paper opens the door for the physical interpretation of all the solutions obtained in this work and hence experimental investigation in the motion of all bodies in the Earth's atmosphere and solar system as well as all other gravitating systems in the universe.

In the second place the door in henceforth opened for the mathematical study of all the unsolved equations of motion for all particles in all types of motions in all systems in the universe, such as more accurate calculation of (i) missile and satellite and space craft trajectories in the earth's atmosphere (ii) motions of moons around their planets, (iii) motion of planets (iv) comets and asteroids around the sun and (v) motions of stars around their galactic nuclei in the universe.

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