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Comparison of Deflection and slope of Cantilever Beam with Analytical and Finite Element Method for Different Loading Conditions

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Abstract— This paper presents the finite element formulation of cantilever beam and obtained deflection and slope and results are compared with Euler–Bernoulli beam theory (analytical solution). Here uniformly varying load (UVL) have been applied, deflection and slope are determined at different sections of beam. The analytical results are different with finite element method solutions for less elements but converges solution when number of elements are increased. The cantilever beam with uniformly distributed loading (UDL) is also analysed with both methods and results are compared to beam with UVL. The finite element method (FEM) and analytical solution are compared for different loadings. Here computer programme in matlab is used to obtain deflection and slope of Euler–Bernoulli beams at different locations for different loadings.

***Index Terms*— Cantilever, Euler–Bernoulli beam, uniformly varying loads, FEM, Matlab**

I. INTRODUCTION

The finite element method (FEM) is useful for many structural problem encountered during engineering application and applied science practical problem. But results obtained through FEM is not reliable like other numerical method. The accuracy of finite element method depends upon the discretization, interpolation function and number of elements. The finite element method consists three important steps: pre-processing, processing and post processing.[3]

In pre-processing, input data i.e. material properties, geometry, loads and boundary conditions are provided before simulation. Model is discretized into elements and meshes are generated automatically through matlab software. In processing stiffness matrix is generated and assembled to global matrix.

Euler-Bernoulli Beam Theory: The assumption considered for Euler- Beam theory is that plane section before and after bending remains plane and perpendicular to axis. The cantilever beam with uniformly varying load (UVL) is shown in the figure (1). Here loading q(x) is varying linearly q_o at fixed end to zero at free end on cantilever beam. At the free end of cantilever the point load (F_o) and moment (M_o) are acting in addition to UVL and UDL in figure (1) and figure (2).The differential equation of fourth order is used as governing equation for beam and transverse deflection is denoted by 'w'[1].

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)=q(x)------(1)$$

Here, 'E' denotes the modulus of elasticity, 'I' is the area moment of inertia of beam about an axis perpendicular to both x and z axis and variation of loading $q(x)=q_o(1-x/L)$ where L is the total length of beam. The cantilever beam with uniformly distributed load (UDL) is shown in the figure (2)

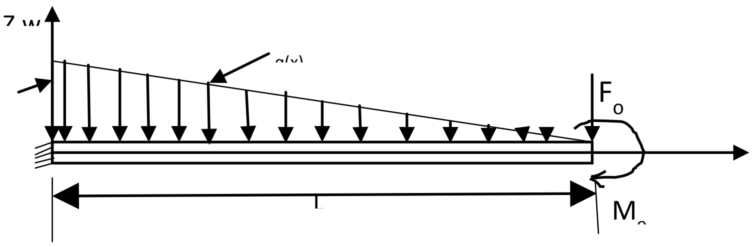

Fig (1) Cantilever beam with Varying distributed load

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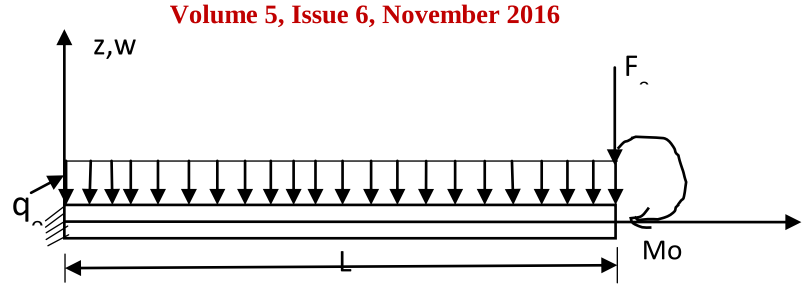

Fig (2) Cantilever beam with uniformly distributed load

The beam consists numbers of elements and considers an n^th element and distance of n^th nodes is at distance 'c' from global coordinates as shown in figure (3).

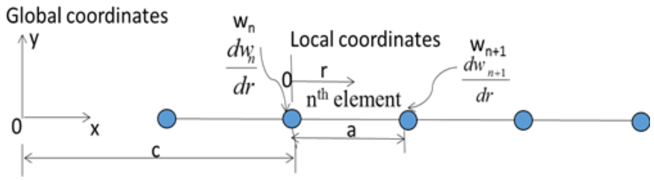

Fig (3) Degree of freedom for n^th element with local and global coordinates.

The transverse displacement (w) for element in z direction is written in terms of four constants c_1, c_2, c_3 and c_4

$$w = c_1 + c_2 r + c_3 r^2 + c_4 r^3 \text{---------------- (2)}$$

Apply end conditions at both ends of element (figure (3)) and four shape functions/interpolations functions (N_1, N_2, N_3&N_4) are obtained as followings. These functions are also known as Hermite cubic interpolation functions.

$$N_1 = \left(1 - 3\frac{r^2}{a^2} + 2\frac{r^3}{a^3}\right), \quad N_2 = \left(r - 2\frac{r^2}{a} + \frac{r^3}{a^2}\right), \quad N_3 = \left(3\frac{r^2}{a^2} - 2\frac{r^3}{a^3}\right), \quad N_4 = \left(\frac{r^3}{a^2} - \frac{r^2}{a}\right) \text{--------(3)}$$

From equation (2), transverse displacement is expressed in terms of interpolation functions.

$$w = N_1 w_n + N_2 \frac{dw_n}{dx} + N_3 w_{n+1} + N_4 \frac{dw_{n+1}}{dx} \text{----(4)}$$

Use Galerkins approach/weighted residual methods to evaluate stiffness matrix and load vector. In this method integrate the weighted error over domain and equated to zero. Here weight function is considered same as 'w'.

$$\int_0^a (ERROR)(w)dr = 0 \text{--------(5)}$$

Element stiffness matrix (Ke) is obtained

$$Ke = \frac{EI}{a^3}\begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} \text{---(6)}$$

The elemental force matrix is obtained by integrating weighted load and obtained as

$$\{f^e\} = \int_0^a q_o\left(1 - \frac{r+c}{L}\right) w(r) dr$$

After integration final elemental load matrix becomes in form of column vector[1]

$$\{f^e\} = \frac{q_o a}{12}\begin{Bmatrix} 6 \\ a \\ 6 \\ -a \end{Bmatrix} - \frac{q_o a}{60L}\begin{Bmatrix} (9a+30c) \\ a(2a+5c) \\ (21a+30c) \\ -a(3a+5c) \end{Bmatrix} \text{---(7)}$$

Here, total number of elements is ten and number of nodes is eleven figure (4).The number within the circle represents the elements and simple numerals on black solid circle represents nodes. From above expression (7), it can be seen that load vector for UVL is not constant for every elements it changes from element to element. Elemental matrix is assembled for whole beam and it becomes global matrix [K]. Similarly elemental load matrix {F} is assembled and global load matrix is obtained.

Fig (4) Number of elements (10) and number of nodes (11)

For UDL the loading q_0 is constant throughout the length of beam, so elemental force matrix is obtained

$$\{f^e\} = \int_0^a q_o w(r) dr$$

$$\{f^e\} = \frac{q_o a}{12} \begin{Bmatrix} 6 \\ a \\ 6 \\ -a \end{Bmatrix} --------(8)$$

If global displacement is represented by { δ }, then by Galerkin method, stiffness, displacement, distributed load and point load are related by relation (8)

$$[K]\{\delta\} = \{F\} + \{F_p\} ----(9)$$

Here F_p is the external point loads acting on the nodes. Apply the boundary conditions at fixed node (node 1) and solution for displacements and rotations are obtained by use of equation (8). Matlab code has been prepared and used to obtain the solution.

The beam is made of steel materials having dimensions and loadings and mechanical properties are following:-length of beam is L=3m, Young modulus E=200x10^6kN/m^2, Moment of inertia about y axis, I=29x$10^{-6}$$m^4$,$q_o$=24kN/m,Point load at free end F_o=60kN,Moment at free end M_o=0kN-m [1].In diagram (2) length and mechanical properties of beam are same but loading conditions are different from figure(1),here uniformly distributed load(UDL) of 24 kN/m is

II. ANALYTICAL METHOD

The deflection of beam for UVL is obtained by direct double integration of moment at section "x" from fixed end as in equation (10). The slope of beam is obtained by direct integration of moment at section "x" from fixed end as in equation (11) [1]. The FEM and analytical solution for displacement and slopes are compared in the diagrams (5 to9).

$$w(x) = \frac{q_o L^4}{120EI}\left[10\left(\frac{x}{L}\right)^2 - 10\left(\frac{x}{L}\right)^3 + 5\left(\frac{x}{L}\right)^4 - \left(\frac{x}{L}\right)^5\right] + \frac{F_o L^3}{6EI}\left[-\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right] + \frac{M_o L^2}{2EI}\left(\frac{x}{L}\right)^2 -------(10)$$

$$\theta(x) = -\frac{q_o L^3}{24EI}\left[4\left(\frac{x}{L}\right) - 6\left(\frac{x}{L}\right)^2 + 4\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^4\right] + \frac{F_0 L^2}{2EI}\left[\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)\right] - \frac{M_0}{EI}\left(\frac{x}{L}\right) ----(11)$$

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The deflection of beam for UDL is obtained by direct double integration of moment at section "x" from fixed end as in equation (12). The slope of beam is obtained by direct integration of moment at section "x" from fixed end as in equation(13) [2].

$$w(x)=\frac{q_oL^4}{24EI}\left[-6\left(\frac{x}{L}\right)^2+4\left(\frac{x}{L}\right)^3-5\left(\frac{x}{L}\right)^4\right]$$

$$+\frac{F_oL^3}{6EI}\left[-\left(\frac{x}{L}\right)^3+3\left(\frac{x}{L}\right)^2\right]+\frac{M_oL^2}{2EI}\left(\frac{x}{L}\right)^2 \quad --(12)$$

$$\theta(x)=\frac{q_oL^3}{6EI}\left[-3\left(\frac{x}{L}\right)+3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right]+\frac{F_oL^2}{2EI}\left[\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)\right]-\frac{M_o}{EI}\left(\frac{x}{L}\right)-------(13)$$

The FEM and analytical solution for UVL loading for entire length of cantilever beam at step of 0.3m is shown in the table (1).Similarly for UDL loading displacement and slope is obtained and shown in table (2).

Table (1) Comparison of finite element method (FEM) solution with analytical for UVL.

Distance from fixed end X(m)	Displacement w(m)		Slope (Θ)	
	FEM	Analytical	FEM	Analytical
0	0	0	0	0
0.3	-0.0016	-0.0016	-0.0104	-0.0104
0.6	-0.0061	-0.0061	-0.0195	-0.0195
0.9	-0.0132	-0.0132	-0.0273	-0.0273
1.2	-0.0224	-0.0224	-0.0338	-0.0338
1.5	-0.0334	-0.0334	-0.0393	-0.0393
1.8	-0.0458	-0.0458	-0.0436	-0.0436
2.1	-0.0595	-0.0595	-0.047	-0.047
2.4	-0.0739	-0.0739	-0.0493	-0.0493
2.7	-0.0890	-0.089	-0.0507	-0.0507
3	-0.1043	-0.1043	-0.0512	-0.0512

Table (2) Comparison of finite element method (FEM) solution with analytical for UDL.

Distance from fixed end X(m)	Displacement w(m)		Slope (Θ)	
	FEM	Analytica l	FEM	Analytical
0	0	0	0	0
0.3	-0.0016	-0.0016	-0.0104	-0.0104
0.6	-0.0061	-0.0061	-0.0195	-0.0195
0.9	-0.0132	-0.0132	-0.0273	-0.0273
1.2	-0.0224	-0.0224	-0.0338	-0.0338
1.5	-0.0334	-0.0334	-0.0393	-0.0393
1.8	-0.0458	-0.0458	-0.0436	-0.0436
2.1	-0.0595	-0.0595	-0.047	-0.047
2.4	-0.0739	-0.0739	-0.0493	-0.0493
2.7	-0.0890	-0.089	-0.0507	-0.0507
3	-0.1043	-0.1043	-0.0512	-0.0512

III. RESULT AND DISCUSSION

Figure (5) and (6) show the deflection and slope of beam respectively due to UVL from analytical and FEM method. The displacement and slope of cantilever beam obtained from analytically matched with finite element solution for 10 elements due for UVL loading (figure (5)&(6)). The data obtained from analytical solution and FEM solution with help of MATLAB code [4]is mentioned in table (1) for UVL loading. First column indicates distance of nodes from fixed end to free of cantilever at step of 0.3m length. Second and third columns indicates value of displacement obtained from FEM and analytical respectively for UVL loading. The slope obtained from FEM and analytically is shown in fourth and fifth column in table(1).

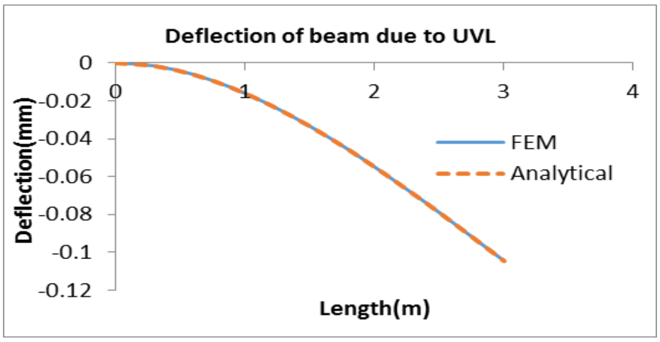

Fig (5): Comparison of deflection obtained analytically and through FEM for UVL.

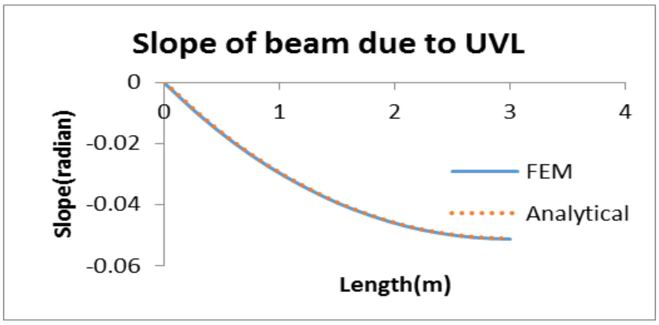

Fig (6): Comparison of slope obtained analytically and through FEM for UVL.

Figure (7) and (8) show the deflection and slope of beam respectively due to UDL from analytical and FEM method with use of MATLAB code [5]. The displacement and slope of cantilever beam obtained from analytically is slightly different with finite element solution for 10 elements due for UVL loading (figure (7)&(8)). This is always true for beam subjected to distributed loading is modeled by cubic displacement function. The solution obtained by beam theory is quartic (fourth order) polynomial but in FEM always it is assumes that displacement is cubical polynomial [2] .So finite element solution predicts stiffer structure than actual structure. The data obtained from analytical solution and FEM solution is mentioned in table (2) for UDL loading. First column indicates distance of nodes from fixed end to free of cantilever at step of 0.3m length. Second and third columns indicates value of displacement obtained from FEM and analytical respectively for UDL loading. The slope obtained from FEM and analytically is shown in fourth and fifth column in table(2).

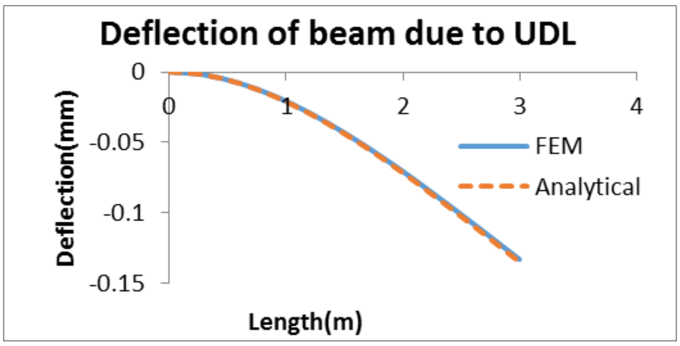

Fig (7): Comparison of deflection obtained analytically and through FEM for UDL.

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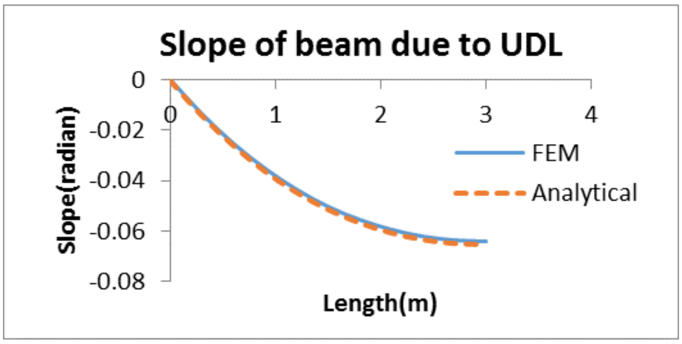

Fig (8): Comparison of slope obtained analytically and through FEM for UDL.

Figure (9) compare the deflection of beam due to UDL and UVL by FEM method for same loading. The displacement of cantilever beam due to UDL are more than UVL. The is true because loading in UDL is uniform thorough out the entire length of beam but in UDL it is maximum at the fixed end of cantilever and decreases to zero at the free end of the beam.

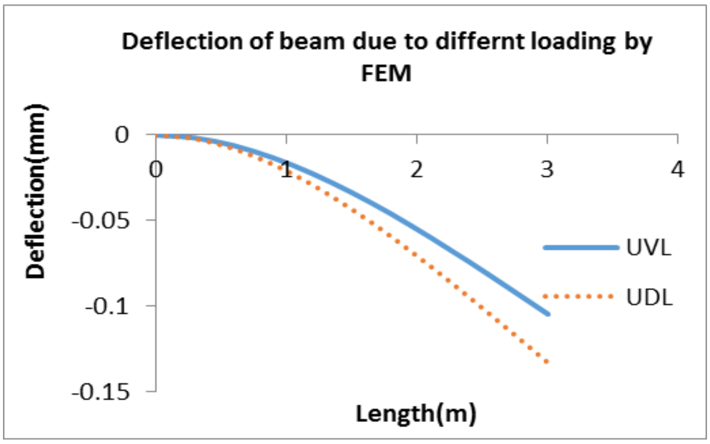

Fig (9): Comparison of deflection for different loadings.

Figure (10) shows slope of beam due to UDL and UVL by FEM method for same loading. The slope of cantilever beam due to UDL are more than UVL. The is true because loading in UDL is uniform thorough out the entire length of beam but in UDL it is maximum at the fixed end of cantilever and decreases to zero at the free end of the beam It is observed that deflection for UDL is greater than the UVL for same loading

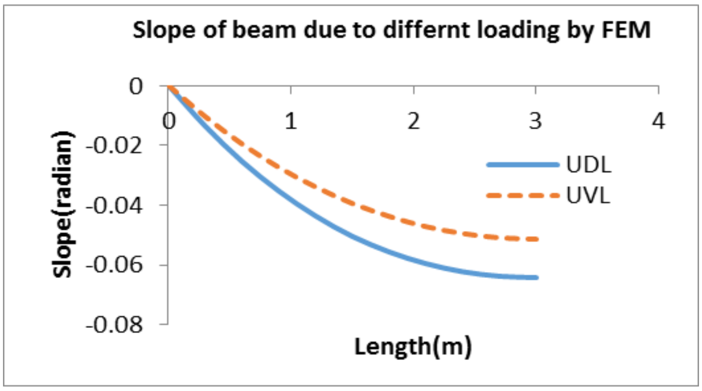

Fig (10): Comparison of slopes for different loadings.

IV. CONCLUSION

Following conclusions are obtained from this research.

- The solution for deflection and slope obtained from FEM and beam theory are exactly matched for ten elements with UVL loading
- The solution for deflection and slope obtained from FEM and beam theory are slightly lower for ten elements with UDL loading.
- Finite element solution predicts stiffer structure than actual structure.
- The deflection and slope are larger for UDL than UVL and increase from fixed end to free end for cantilever beam.

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