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Some approximate properties of Cesaro means Fourier series and their conjugates

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ABSTRACT. It is established $\sigma_n^a(x,f)$ ***and*** $t_n^a(x,f)$ ***Cesaro means some approximate features. AMS Subject Classification: 20M05***

Key words and phrases: **Chesaro means, Fourier trigonometric series, conjugated trigonometric series.**

I. INTRODUCTION

Suppose, that $T=[-\pi,\pi]$ and $f:R\to R$ 2π - are periodic functions. If $f\in L(T)$ as a rule, $\sigma(f)$ and $\bar{\sigma}(f)$ represent respectively trigonometric Furrier series and their conjugates

$$\sigma[f](x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx+b_k\sin kx, \qquad (1) \qquad \bar{\sigma}[f](x)=\sum_{k=1}^{\infty}-b_k\cos kx+a_k\sin kx, \qquad (2)$$

where

$$\left.\begin{aligned} a_k\equiv a_k(f)&=\frac{1}{\pi}\int_T f(t)\cos kt\,dt, \quad k\in N_0\\ b_k=b_k(f)&=\frac{1}{\pi}\int_T f(t)\sin kt\,dt, \quad k\in N.\end{aligned}\right\} . \qquad (3)\text{-}(4)$$

As it is known, the issue of convergence and summability of series $\bar{\sigma}[f](x)$ are closely related with corresponding properties of conjugated function $\bar{f}$, which is defined with the following equation

As I. I. Privalov showed [1], in $f\in L(T)$ function $\bar{f}$ exists almost everywhere. Let us assign

$$\bar{f}_n(x)=-\frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\pi}[f(x+t)-f(x-t)]\mathrm{ctg}\frac{t}{2}dt, \quad n>1. \qquad (5)$$

Assume that $p\in[1,+\infty[$ - is a number. For each function $f\in L^p(T)$ the following will be considered

$$\|f\|_p=\left\{\frac{1}{2\pi}\int_T|f(t)|^p dt\right\}^{\frac{1}{p}},$$

And also it will consider the following: $L^\infty(T)=c(T)$, $\|f\|_c=\|f\|_\infty=\sup_{x\in T}|f(x)|$. Let us set

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$$\omega^{(k)}(\sigma, f) = \sup_{|h|\le\sigma} \left\| \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} f(t+jh) \right\|_p \; ; \quad \sigma \in]0, 2\pi]$$

$\omega^{(k)}(\sigma, f)$ is called module L^p of smoothness of an order to function f.

In the future we will assume that $\omega^{(1)}(\sigma, f)_p \equiv \omega(\sigma, f)_p$.

Bellow A, $A(f),\ A(f,p),\ A(f,a,p),\ A(a),\ A_1(a), \ldots$ indicate absolute positive or positive constants depending only on the specified parameters.

Let ω-be module of continuity. Assume that

$H_p^\omega \equiv H_p^\omega(T) = \{ f : \omega(\sigma, f)_p \le A(f,p)\omega(\sigma) \}$ and $H_p^\omega = H^\omega$. If $\omega(\sigma) = \sigma^\alpha,\ \alpha \in]0,1]$, than $H_p^\omega \equiv Lip(a,p).\ H^\omega \equiv Lipa.$

With $S_n(x,f)$ and $\bar{S}_n(x,f)$ we will define, respectively partial sums of the series

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin ks = \frac{1}{\pi} \int_T f(x+t) D_n(t) dt, \qquad (6)$$

$$\bar{S}_n(x,f) = \sum_{k=1}^{n} - b_k \cos kx + a_k \sin kx = -\frac{1}{\pi} \int_T f(x+t) \bar{D}_n(t) dt, \qquad (7)$$

where $D_n(t)$- is Dirichlet kernel, and $\bar{D}_n(t)$- conjugate kernel, i.e.

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}}, \qquad (8)$$

$$\bar{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} \quad t \in]0, \pi], \qquad (9)$$

and $Dn(0) = n + \frac{1}{2},\ \ \bar{D}_n(0) = 0,\ \ n \in N.$

Consider that $A_k^a = 1,\ A_k^a = \dfrac{(a+1)(a+2)(a+3)\ldots.(a+k)}{k!},\quad k \in N,\ a > -1. \qquad (10)$

It is known (see e.g. A. Sigmund [7], pp. 130-131) that $A_k^a = \sum_{i=0}^{k} A_{k-i}^{a-1},\ \ A_k^a - A_{k-1}^a = A_k^{a-1}$ (11), $A(a) \le \dfrac{A_k^a}{k^a} \le A_1(a)$ (12) ,

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If $K_n^a(t)=\frac{1}{A_n^a}\sum_{k=0}^{n}A_{n-k}^{a-1}D_k(t)$, (13), $\tau_n^a(t)=\frac{1}{A_n^a}\sum_{k=0}^{n}A_{n-k}^{a-1}\bar{D}_k(t)$, (14)

than they are respectively called Chesaro kernel and its conjugate kernel. It is known that (see e.g. A. Sigmund [2] pp. 157-164),

that $K_n^a(t)=\varphi_n^a(t)+r_n^a(t)$, (15) $r_n^a(t)=\frac{1}{2}\operatorname{ctg}\frac{t}{2}+\psi_n^a(t)+\gamma_n^a(t),\ t\in]0.\pi]$ (16)

where $\varphi_n^a(t)=\frac{\sin\left[\left(n+\frac{1}{2}+\frac{a}{2}\right)t-\frac{a\pi}{2}\right]}{A_n^a\left(2\sin\frac{t}{2}\right)^{1+a}}$ (17), $\psi_n^a(t)=\frac{\cos\left[\left(n+\frac{1}{2}+\frac{a}{2}\right)t-\frac{a\pi}{2}\right]}{A_n^a\left(2\sin\frac{t}{2}\right)^{1+a}}$ (18),

and $\|K_n^a\|_c\le A(a)n$, (19), $\|\tau_n^a\|_c\le A(a)n$, (20), $|\tau_n^a(t)|\le\frac{A(a)}{nt^2},\ \frac{\pi}{n}\le t\le\pi$, (21),

$$|\gamma_n^a(t)|\le\frac{A(a)}{nt^2},\ \frac{\pi}{n}\le t\le\pi. \quad (22)$$

In what follows we shall use the Holder inequality for integrals. If $f_1\in L^p(T),\ f_2\in L^q(T)$ и $\frac{1}{p}+\frac{1}{q}=1,$ than

$$\|f_1 f_2\|_L\le\|f_1\|_p\|f_2\|_q \quad . \quad (23)$$

We also use the Minkowski inequality. If $f_1\in L^p(T),\ f_2\in L^p(T),\ p\in[1,+\infty[$, than

$$\|f_1+f_2\|_p\le\|f_1\|_p+\|f_1\|_p \quad (24)$$

(see, e.g., .Hardy, Littlewood, Polia, p. 179 [3]) the generalized Minkowski inequality is true (under appropriate conditions). If $p\in[1,+\infty[$ is a number, than

$$\left\{\int_T\left[\int_T|f(x_1,x_2)dx_1|^p\right]dx_2\right\}^{\frac{1}{p}}\le\int_T\left\{\int_T|f(x_1,x_2)|^p\,dx_2\right\}^{\frac{1}{p}}dx_1. \quad (25)$$

Assume that function $f\in L(T)$ than $\sigma_n^a(x,f)$ and $t_n^a(x,f)$ symbols denotes the Chesaro means of order $a>-1$ consequently $\sigma[f]$ and $\bar{\sigma}[f]$ i.e.

$$\sigma_n^a(x,f)=\frac{1}{A_n^a}\sum_{k=0}^{n}A_{n-1}^{a-1}S_k(x,f),\qquad t_n^a(x,f)=\frac{1}{A_n^a}\sum_{k=0}^{n}A_{n-1}^{a-1}\bar{S}_k(x,f)$$

Where $S_k(x,f)$ and $\bar{S}_k(x,f)$ are given to relations (6) and (7). Using the equalities (13) and (14), we can write

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$$\sigma_n^a(x,f)=\frac{1}{\pi}\int_T f(x+t)K_n^a(t)dt \quad (26), \qquad t_n^a(x,f)=\frac{1}{\pi}\int_T f(x+t)\tau_n^a(t)dt \quad (27).$$

Different issues related to the actions $\sigma_n^a(x,f)$ and $t_n^a(x,f)$ averages, are studies in the monograph L.V.Zhizhiashvili. ([4], volumes II-IV).

In this paper we establish some new properties $\sigma_n^a(x,f)$ и $t_n^a(x,f)$. Further assume that

$$g(n,f)=\frac{1}{n}\int_{\frac{1}{n}}^{\pi}\frac{\omega(t,f)_c}{t^2}dt \quad (28); \qquad \text{Here we note that } \omega\left(\frac{1}{n},f\right)\le g(n,f) \quad . \quad (29)$$

Theorem 1. Let $a\in]0,1]$ - a curtain number $ap=1,\ \beta\in]0,a[$ и $0<\lambda<\frac{1}{2}$

a) If $\lambda\ge\frac{1}{1+a}$ than for the function $f\in c(T)$ the following inequality is true

$$\left\|\sigma_n^{-\beta}(f)-f\right\|_c\le A(a,\beta)\left[\omega^{\lambda(a-\beta)}\left(\frac{1}{n},f\right)_c\right]+\left[1+n^a\omega\left(\frac{1}{n},f\right)_p\right]+A(\beta)g(n,f)$$

b) And if $\lambda<\frac{1}{1+a}$, than for the function $f\in c(T)$ the following inequality is true

$$\left\|\sigma_n^{-\beta}(f)-f\right\|_c\le A(a,\beta)\left[\omega^{1-\lambda(a+\beta)}\left(\frac{1}{n},f\right)_c+n^a\omega\left(\frac{1}{n},f\right)_p\omega^{\lambda(a-\beta)}\left(\frac{1}{n},f\right)_c\right]+A(\beta)g(n,f),\quad n\ge 4.$$

Proof. We have

$$\sigma_n^{-\beta}(x,f)-f(x)=\frac{1}{\pi}\int_0^{\frac{\pi}{n}}\varphi(x,t)K_n^{-\beta}(t)dt+\frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\varphi(x,t)\varphi_n^{-\beta}(t)dt+\frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi}\varphi(x,t)r_n^{-\beta}(t)dt\equiv\sum_{i=1}^{3}B_n^{(i)}(x,f,\beta) \quad (30)$$

When analyzing the scheme of the proof of Theorem 1(M.S.Bakuridze [5]), we conclude that it is sufficient to establish a proper assessment for $B_n^{(2)}(x,f,\beta)$ from (30)

As indicated in (38) (M.S.Bakuridze [5]), it is sufficient to evaluate the following integral

$$B_n^{(4)}(x,f,\beta)=n^{\beta}\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\left[\varphi(x,t)-\varphi\left(x,t+\frac{\pi}{n}\right)\right]V_{\beta}(t)\sin ntdt$$

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Assume that $b(n,f,\lambda)=\frac{\pi}{\omega^{\lambda}\left(\frac{1}{n},f\right)_c}$, than we will have

$$B_n^{(4)}(x,f,\beta)=n^{\beta}\int_{\frac{\pi}{n}}^{\frac{b(n,f,\lambda)}{n}}\left[\varphi(x,t)-\varphi\left(x,t+\frac{\pi}{n}\right)\right]V_{\beta}(t)\sin ntdt+$$

$$+n^{\beta}\int_{\frac{b(n,f,\lambda)}{n}}^{\pi-\frac{\pi}{n}}\left[\varphi(x,t)-\varphi\left(x,t+\frac{\pi}{n}\right)\right]V_{\beta}(t)\sin ntdt\equiv B_n^{(5)}(x,f,\beta,\lambda)+B_n^{(6)}(x,f,\beta,\lambda) \quad (31)$$

From this we can conclude that

$$\left\|B_n^{(5)}(x,f,\beta,\lambda)\right\|_c\le A(\beta)n^{\beta}\omega\left(\frac{1}{n},f\right)_c\left[\frac{1}{\omega^{\lambda}\left(\frac{1}{n},f\right)_c}+1\right]\int_{\frac{\pi}{n}}^{\frac{b(x,f,\lambda)}{n}}\frac{dt}{t^{1-\beta}}\le A(\beta)\omega^{1-\lambda(1+\beta)}\left(\frac{1}{n},f\right)_c \quad (32)$$

By using Hölder's inequality $\left(p=\frac{1}{a},\ q=\frac{1}{1-a}\right)$, from the equality (59) we conclude that

$$\left\|B_n^{(6)}(.,f,\beta,\lambda)\right\|_c\le A(\beta)n^{\beta}\left\{\int_T\left|f\left(t+\frac{\pi}{n}\right)-f(t)\right|^{\frac{1}{a}}dt\right\}^{a}\left\{\int_{\frac{b(x,\beta,\lambda)}{n}}^{\pi}\frac{dt}{t^{1-\beta(1-a)}}\right\}^{1-a}\le$$

$$\le A(a,\beta)n^{\beta}\omega\left(\frac{1}{n},f\right)_{1/a}n^{a-\beta}\omega^{\lambda(a-\beta)}\left(\frac{1}{n},f\right)_c=A(a,\beta)n^{a}\omega\left(\frac{1}{n},f\right)_{1/a}\omega^{\lambda(a-\beta)}\left(\frac{1}{n},f\right)_c \quad (33)$$

Thus, the relations (30) - (33) make it possible to conclude that Theorem 2 is true.

Now consider the average $t_t^{a}(x,f)$ of series $\sigma[f]$. If we take into account (5), (16) and (27), we will find

$$t_t^{-a}(x,f)-\bar{f}_n(x)=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}[f(x+t)-f(x-t)]\tau_n^{-a}(t)dt+$$

$$+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}[f(x+t)-f(x-t)]\psi_n^{-a}(t)dt+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}[f(x+t)-f(x-t)]\gamma_n^{-a}(t)dt\equiv \quad (34)$$

$$\equiv I_n^{(1)}(x,f,a)+I_n^{(2)}(x,f,a)+I_n^{(3)}(x,f,a)$$

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If we take into account (18, (20) and (22) and the proof of the theorems 1 and 2 (M.S.Bakuridze [5]), , we will conclude that in (34) expression $I_n^{(i)}(x, f, a)$ $(i = 1,2,3)$ can be evaluated by the same values that we have in Theorems 1 and 2.

Consequently, we have the following theorems

Theorem 2. a) Let $a \in]0,1[$ - a curtain number. If the function $f \in C(T), a, ap > 1$ than

$$\left\| t_n^{-a}(f) - \bar{f}_n \right\|_c \le A(p,a) n^a \omega\left(\frac{1}{n}, f\right)_p + A(a) g(n, f)$$

b) Let $p \in]1, \infty[$ - a curtain number, and function $f \in C(T)$. Than

$$\left\| t_n^{-\frac{1}{p}}(f) - \bar{f}_n \right\|_c \le A(p) \left[n^{\frac{1}{p}} (\ln n)^{1-1/p} \omega\left(\frac{1}{n}, f\right)_p + g(n, f) \right].$$

c) Suppose that $p \in]0, +\infty[$ - is a curtain number and function $f \in C(T)$. If $ap \in]0,1[$ than

$$\left\| t_n^{-a}(f) - \bar{f}_n \right\|_c \le A(p,a) n^{\frac{1}{p}} \omega\left(\frac{1}{n}, f\right)_p + A(a) g(n, f), \quad n \ge 4$$

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