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Some approximate properties of Cesaro means Fourier series and their conjugates

Mzevinar Bakuridze
Faculty of Physics, Mathematics and Computer Sciences, Department of Mathematics,
Shota Rustaveli Batumi State University, 35 Ninoshvili St. Batumi 6010, Georgia

ABSTRACT: *It is established* $\sigma_n^a(x,f)$ *and* $t_n^a(x,f)$ *Cesaro means some approximate features.*

Key words and phrases**: Chesaro means, Fourier trigonometric series, conjugated trigonometric series, AMS Subject Classification, 20M05.**

I. INTRODUCTION

Suppose, that $T=[-\pi,\pi]$ and $f:R\to R$ 2π - are periodic functions. If $f\in L(T)$ as a rule, $\sigma(f)$ and $\bar{\sigma}(f)$ represent respectively trigonometric Furrier series and their conjugates

$$\sigma[f](x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx+b_k\sin kx, \quad (1) \qquad \bar{\sigma}[f](x)=\sum_{k=1}^{\infty}-b_k\cos kx+a_k\sin kx, \quad (2)$$

where

$$\left.\begin{aligned} a_k\equiv a_k(f)=\frac{1}{\pi}\int_T f(t)\cos kt\,dt, \quad k\in N_0 \\ b_k=b_k(f)=\frac{1}{\pi}\int_T f(t)\sin kt\,dt, \quad k\in N. \end{aligned}\right\} . \qquad (3)$$

As it is known, the issue of convergence and summability of series $\bar{\sigma}[f](x)$ are closely related with corresponding properties of conjugated function $\bar{f}$, which is defined with the following equation:

As I. I. Privalov showed [1], in $f\in L(T)$ function $\bar{f}$ exists almost everywhere. Let us assign

$$\bar{f}_n(x)=-\frac{1}{2\pi}\int_{\frac{\pi}{n}}^{\pi}[f(x+t)-f(x-t)]\,ctg\frac{t}{2}\,dt, \quad n>1. \qquad (5)$$

Assume that $p\in[1,+\infty[$ - is a number. For each function $f\in L^p(T)$ the following will be considered

$$\|f\|_p=\left\{\frac{1}{2\pi}\int_T |f(t)|^p\,dt\right\}^{\frac{1}{p}},$$

And also it will consider the following: $L^\infty(T)=c(T)$, $\|f\|_c=\|f\|_\infty=\sup_{x\in T}|f(x)|$. Let us set

$$\omega^{(k)}(\sigma,f)=\sup_{|h|\le\sigma}\left\|\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}f(t+jh)\right\|_p ; \quad \sigma\in]0,2\pi].$$

$\omega^{(k)}(\sigma,f)$ is called module L^p of smoothness of an order to function f.

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In the future we will assume that $\omega^{(1)}(\sigma, f)_p \equiv \omega(\sigma, f)_p$.

Bellow A, $A(f),\ A(f,p),\ A(f,a,p),\ A(a),\ A_1(a), \ldots$ indicate absolute positive or positive constants depending only on the specified parameters.

Let ω -be module of continuity. Assume that

$H_p^{\omega} \equiv H_p^{\omega}(T) = \{f : \omega(\sigma, f)_p \le A(f,p)\omega(\sigma)\}$ and $H_p^{\omega} = H^{\omega}$. If $\omega(\sigma) = \sigma^{\alpha},\ \ \alpha \in]0,1]$, than $H_p^{\omega} \equiv Lip(a,p).\ \ H^{\omega} \equiv Lipa.$

With $S_n(x,f)$ and $\bar{S}_n(x,f)$ we will define, respectively partial sums of the series

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin ks = \frac{1}{\pi}\int_T f(x+t)D_n(t)dt, \qquad (6)$$

$$\bar{S}_n(x,f) = \sum_{k=1}^{n} -b_k \cos kx + a_k \sin kx = -\frac{1}{\pi}\int_T f(x+t)\bar{D}_n(t)dt, \qquad (7)$$

where $D_n(t)$ - is Dirichlet kernel, and $\bar{D}_n(t)$ - conjugate kernel, i.e.

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}, \qquad (8)$$

$$\bar{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} \quad t \in]0,\ \pi], \qquad (9)$$

and $Dn(0) = n + \frac{1}{2},\ \ \bar{D}_n(0) = 0,\ \ n \in N.$

Consider that $\quad A_k^a = 1,\ A_k^a = \frac{(a+1)(a+2)(a+3).....(a+k)}{k!},\quad k \in N,\ a > -1. \qquad (10)$

It is known (see e.g. A. Sigmund [7], pp. 130-131) that $A_k^a = \sum_{i=0}^{k} A_{k-i}^{a-1},\ \ A_k^a - A_{k-1}^a = A_k^{a-1}$ (11), $A(a) \le \frac{A_k^a}{k^a} \le A_1(a)$ (12) ,

If $K_n^a(t) = \frac{1}{A_n^a}\sum_{k=0}^{n} A_{n-k}^{a-1} D_k(t),$ (13), $\quad \tau_n^a(t) = \frac{1}{A_n^a}\sum_{k=0}^{n} A_{n-k}^{a-1}\bar{D}_k(t),$ (14)

than they are respectively called Chesaro kernel and its conjugate kernel. It is known that (see e.g. A. Sigmund [2] pp. 157-164), that $K_n^a(t) = \varphi_n^a(t) + r_n^a(t),$ (15) $\quad r_n^a(t) = \frac{1}{2}\text{ctg}\frac{t}{2} + \psi_n^a(t) + \gamma_n^a(t),\ \ t \in]0.\pi]$ (16)

where $\varphi_n^a(t) = \frac{\sin\left[\left(n+\frac{1}{2}+\frac{a}{2}\right)t - \frac{a\pi}{2}\right]}{A_n^a\left(2\sin\frac{t}{2}\right)^{1+a}}$ (17), $\quad \psi_n^a(t) = \frac{\cos\left[\left(n+\frac{1}{2}+\frac{a}{2}\right)t - \frac{a\pi}{2}\right]}{A_n^a\left(2\sin\frac{t}{2}\right)^{1+a}}$ (18), and

$\|K_n^a\|_c \le A(a)n,$ (19), $\|\tau_n^a\|_c \le A(a)n,$ (20), $|\tau_n^a(t)| \le \frac{A(a)}{nt^2},\ \frac{\pi}{n} \le t \le \pi,$ (21), $|\gamma_n^a(t)| \le \frac{A(a)}{nt^2},\ \frac{\pi}{n} \le t \le \pi.$ (22)

In what follows we shall use the Holder inequality for integrals. If $f_1 \in L^p(T),\ f_2 \in L^q(T)$ и $\frac{1}{p}+\frac{1}{q}=1,$ than

$$\| f_1 f_2 \|_L \le \| f_1 \|_p \| f_2 \|_q \quad . \qquad (23)$$

We also use the Minkowski inequality. If $f_1 \in L^p(T)$, $f_2 \in L^p(T)$, $p \in [1,+\infty[$, than

$$\| f_1 + f_2 \|_p \le \| f_1 \|_p + \| f_1 \|_p \quad (24)$$

(see, e.g., .Hardy, Littlewood, Polia, p. 179 [3]) the generalized Minkowski inequality is true (under appropriate conditions). If $p \in [1,+\infty[$ is a number, than

$$\left\{ \int_T \left[\int_T |f(x_1, x_2) dx_1|^p \right] dx_2 \right\}^{\frac{1}{p}} \le \int_T \left\{ \int_T |f(x_1, x_2)|^p \, dx_2 \right\}^{\frac{1}{p}} dx_1 . \qquad (25)$$

Assume that function $f \in L(T)$ than $\sigma_n^a(x,f)$ and $t_n^a(x,f)$ symbols denotes the Chesaro means of order $a > -1$ consequently $\sigma[f]$ and $\bar{\sigma}[f]$ i.e.

$$\sigma_n^a(x,f) = \frac{1}{A_n^a} \sum_{k=0}^{n} A_{n-1}^{a-1} S_k(x,f), \qquad t_n^a(x,f) = \frac{1}{A_n^a} \sum_{k=0}^{n} A_{n-1}^{a-1} \bar{S}_k(x,f)$$

where $S_k(x,f)$ and $\bar{S}_k(x,f)$ are given to relations (6) and (7). Using the equalities (13) and (14), we can write

$$\sigma_n^a(x,f) = \frac{1}{\pi} \int_T f(x+t) K_n^a(t) dt \qquad (26), \qquad t_n^a(x,f) = \frac{1}{\pi} \int_T f(x+t) \tau_n^a(t) dt \qquad (27).$$

Different issues related to the actions $\sigma_n^a(x,f)$ and $t_n^a(x,f)$ averages, are studies in the monograph L.V.Zhizhiashvili. ([4], volumes II-IV).
In this paper we establish some new properties $\sigma_n^a(x,f)$ и $t_n^a(x,f)$. Further assume that

$$g(n,f) = \frac{1}{n} \int_{\frac{1}{n}}^{\pi} \frac{\omega(t,f)_c}{t^2} dt \qquad (28); \qquad \text{Here we note that } \omega\left(\frac{1}{n}, f\right) \le g(n, f) \ . \quad (29)$$

<u>Theorem 1</u>. a) Let $a \in (0, 1)$, $p \in (1, +\infty)$ and $ap > 1$. Than for the function $f \in C(T)$ the following inequality is true

$$\left\| \sigma_n^{-a}(f) - f \right\|_c \le A(p,a) n^a \omega\left(\frac{1}{n}, f\right)_p + A(a) g(n,f)$$

b) Let $p \in (1, +\infty)$. Than for the function $f \in C(T)$ the following inequality is true

$$\left\| \sigma_n^{-\frac{1}{p}}(f) - f \right\|_c \le A(p) \left[n^{\frac{1}{p}} (\ln n)^{1-\frac{1}{p}} \omega\left(\frac{1}{n}, f\right)_p + g(n,f) \right]$$

c) Let $p \in (1, +\infty)$, $a \in (0, 1)$ and $ap \in]0,1[$ than for the function $f \in C(T)$ the following inequality is true

$$\left\| \sigma_n^{-a}(f) - f \right\|_c \le A(p,a) n^{\frac{1}{p}} \omega\left(\frac{1}{n}, f\right)_p + A(a) g(n,f); .$$

<u>Proof.</u>
We will be guided by the method used in the monograph L.V.Zhizhiashvili [4],

Taking into account the relations (8), (11) and (13), conclude that o $\frac{2}{\pi} \int_0^{\pi} K_n^{-a}(t) dt = 1$. Therefore, according to equality (26) we find

$$\sigma_n^{-a}(x,f) - f(x) = \frac{1}{\pi} \int_0^{\pi} \varphi(x,t) K_n^{-a}(t) dt \qquad (30),$$

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where $\varphi(x,t) = f(x+t) + f(x-t) - 2f(x)$. (31)

By force of (15) and (30) we will have

$$\sigma_n^{-a}(x,f) - f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{n}} \varphi(x,t) K_n^{-a}(t)\,dt + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \varphi(x,t)\varphi_n^{-a}(t)\,dt + \frac{1}{\pi}\int_{\frac{\pi}{n}}^{\pi} \varphi(x,t) r_n^{-a}(t)\,dt \equiv U_n^{(1)}(x,f,a) + U_n^{(2)}(x,f,a) + U_n^{(3)}(x,f,a).$$

(32)

From which, on the basis of inequality (19) we conclude

$$\left\| U_n^{(1)}(.,f,\alpha) \right\|_c \le A(a)\,\omega\left(\frac{1}{n}, f\right)_c . \qquad (33)$$

Further, from the equality (30) considering the relation (21) and symbol (28), we will have

$$\left\| U_n^{(3)}(.,f,\alpha) \right\|_c \le A(a)\,g(n,f) . \qquad (34)$$

As

$$\sin\left[\left(n + \frac{1}{2} - \frac{a}{2}\right)t - \frac{a\pi}{2}\right] = \sin\left(n + \frac{1}{2} - \frac{a}{2}\right)t \cos\frac{a\pi}{2} - \cos\left(n + \frac{1}{2} - \frac{a}{2}\right)t \sin\frac{a\pi}{2}$$

$$\sin\left(n + \frac{1}{2} - \frac{a}{2}\right)t = \sin nt\, \cos\left(\frac{1}{2} - \frac{a}{2}\right)t + \cos nt\, \sin\left(\frac{1}{2} - \frac{a}{2}\right)t,$$

$$\cos\left(n + \frac{1}{2} - \frac{a}{2}\right)t = \cos nt \cos\left(\frac{1}{2} - \frac{a}{2}\right)t - \sin nt \sin\left(\frac{1}{2} - \frac{a}{2}\right)t$$

Then by force of the relations (12) and (17), from the equality (32), that the evaluation of the expression $U_n^{(2)}(x,f,a)$ it is sufficient to estimate

$$U_n^{(4)}(x,f,a) = n^a \int_{\frac{\pi}{n}}^{\pi} \varphi(x,t) \frac{\cos\left(\frac{1}{2} - \frac{a}{2}\right)t}{\left(\sin\frac{t}{2}\right)^{1-a}} \sin nt\,dt \qquad (35),$$

$$U_n^{(5)}(x,f,a) = n^a \int_{\frac{\pi}{n}}^{\pi} \varphi(x,t) \frac{\cos\left(\frac{1}{2} - \frac{a}{2}\right)t}{\left(\sin\frac{t}{2}\right)^{1-a}} \cos nt\,dt \qquad (36)$$

Let's consider expression $U_n^{(4)}(x,f,a)$ and consider that

$$V_a(t) \equiv \frac{\cos\left(\frac{1}{2} - \frac{a}{2}\right)t}{\left(\sin\frac{t}{2}\right)^{1-a}} \qquad (37)$$

If in the expression $U_n^{(4)}(x,f,a)$, we replace t with $t + \frac{\pi}{n}$ and summarize the received two values $U_n^{(4)}(x,f,a)$ we will have

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$$2U_n^{(4)}(x,f,a) = n^a\left\{\int_{\frac{\pi}{n}}^{\pi}\varphi(x,t)V_a(t)\sin nt dt - \int_0^{\pi-\frac{\pi}{n}}\varphi\left(x,t+\frac{\pi}{n}\right)V_a\left(t+\frac{\pi}{n}\right)\sin nt dt\right\} =$$

$$= n^a\left\{\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\left[\varphi(x,t)-\varphi\left(x,t+\frac{\pi}{n}\right)\right]V_a(t)\sin nt dt + \int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\varphi\left(x,t+\frac{\pi}{n}\right)\left[V_a(t)-V_a\left(t+\frac{\pi}{n}\right)\right]\sin nt -\right.$$

$$\left.-\int_0^{\frac{\pi}{n}}\varphi\left(x,\ t+\frac{\pi}{n}\right)V_a\left(t+\frac{\pi}{n}\right)\sin nt dt + \int_{\pi-\frac{\pi}{n}}^{\pi}\varphi(x,t)V_a(t)\sin nt dt\right\} \equiv \sum_{i=6}^{9}U_n^{(i)}(x,f,a) \quad . \qquad (38)$$

Using Helder's inequality, according to the equality (38), we find

$$\left\|U_n^{(6)}(\cdot,f,a)\right\|_C \le n^a\left\{\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\left|\varphi(x,t)-\varphi\left(x,t+\frac{\pi}{n}\right)\right|^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\left|V_a(t)\right|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}} \quad . \qquad (39)$$

If we take into account expression (37) and the fact that $\sin t \ge \frac{2}{\pi}t, \quad t \in \left]0,\frac{\pi}{2}\right]$ (40).

We find the following

$$\left\{\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\left|V_a(t)\right|^{\frac{p}{p-1}}dt\right\}^{\frac{p-1}{p}} \le A(a)\left\{\int_{\frac{\pi}{n}}^{\pi} t^{-(1-a)\frac{p}{p-1}}dt\right\}^{\frac{p-1}{p}} \le A(p,a)\left(\pi - n^{\frac{1}{p}-a}\right), \quad \text{if} \quad ap>1 \qquad (41)$$

$$\left\{\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\left|V_a(t)\right|^{\frac{p}{p-1}}dt\right\}^{\frac{p-1}{p}} \le A(p)(\ln n)^{\frac{p-1}{p}}, \quad \text{если} \quad ap=1 \qquad (42)$$

and

$$\left\{\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}\left|V_a(t)\right|^{\frac{p}{p-1}}dt\right\}^{\frac{p-1}{p}} \le A(p,a)\left(n^{\frac{1}{p}-a}\right), \quad \text{если} \quad a<\frac{1}{p} \qquad (43)$$

than, using the relations (39), (41)-(43), we find

$$\left\|U_n^{(6)}(\cdot,f,a)\right\|_C \le \begin{cases} A(p,a)n^a\omega\left(\frac{1}{n},f\right)_p, & ap>1 \\ A(p)n^{\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}\omega\left(\frac{1}{n},f\right)_p, & ap=1 \\ A(p,a)n^{\frac{1}{p}}\omega\left(\frac{1}{n},f\right)_p, & ap<1 \end{cases} \quad . \qquad (44)$$

Consider expression $U_n^{(7)}(x,f,a)$ from the equality (38). We have

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$$U_n^{(7)}(x,f,a)=n^a\left\{\int_{\frac{\pi}{2n}}^{\frac{2\pi}{n}}\varphi\left(x,\ t+\frac{\pi}{n}\right)\left[V_a(t)-V_a\left(t+\frac{\pi}{n}\right)\right]\sin dt+\int_{\frac{2\pi}{n}}^{\pi-\frac{\pi}{n}}\varphi\left(x,\ t+\frac{\pi}{n}\right)\left[V_a(t)-V_a\left(t+\frac{\pi}{n}\right)\right]\sin dt\right\}.$$

Again by the transformation of the variable t we have

$$\begin{aligned}U_n^{(7)}(x,f,a)=n^a\Bigg\{&\int_{\frac{\pi}{n}}^{\frac{2\pi}{n}}\varphi\left(x,\ t+\frac{\pi}{n}\right)\left[V_a(t)-V_a\left(t+\frac{\pi}{n}\right)\right]\sin ntdt+\\&+\frac{n^a}{2}\int_{\frac{2\pi}{n}}^{\pi-\frac{\pi}{n}}\left[\varphi\left(x,\ t+\frac{\pi}{n}\right)-\varphi\left(x,\ t+\frac{2\pi}{n}\right)\right]\left[V_a(t)-V_a\left(t+\frac{\pi}{n}\right)\right]\sin ntdt+\\&\frac{n^a}{2}\int_{\frac{2\pi}{n}}^{\pi-\frac{\pi}{n}}\varphi\left(x,\ t+\frac{2\pi}{n}\right)\left[V_a(t)-2V_a\left(t+\frac{\pi}{n}\right)+V_a\left(t+\frac{2\pi}{n}\right)\right]\sin ntdt-\\&-\frac{n^a}{2}\int_{\frac{2\pi}{n}}^{\frac{2\pi}{n}}\varphi\left(x,\ t+\frac{2\pi}{n}\right)\left[V_a\left(t+\frac{\pi}{n}\right)-V_a\left(t+\frac{2\pi}{n}\right)\right]\sin ntdt+\\&+\frac{n^a}{2}\int_{\pi-\frac{2\pi}{n}}^{\pi-\frac{\pi}{n}}\varphi\left(x,\ t+\frac{2\pi}{n}\right)\left[V_a(t)-V_a\left(t+\frac{\pi}{n}\right)\right]\sin ntdt\Bigg\}\equiv\sum_{i=8}^{12}U_n^{(i)}(x,f,a)\end{aligned}\qquad(45)$$

It is known that

$$\left|V_a(t)-V_a\left(t+\frac{\pi}{n}\right)\right|\le\frac{A(a)}{nt^{2-a}},\quad\frac{\pi}{n}\le t\le\pi\qquad(46)$$

$$\left|V_a(t)-2V_a\left(t+\frac{\pi}{n}\right)+V_a\left(t+\frac{2\pi}{n}\right)\right|\le\frac{A(a)}{n^2t^{3-a}},\quad\frac{\pi}{n}\le t\le\pi\quad.\qquad(47)$$

Considering the inequalities (46) and (47), from (45) we will have

$$\left\|U_n^{(8)}(.,f,a)\right\|_c\le A(a)\omega\left(\frac{1}{n},f\right)_c,\qquad(48)$$

$$\left\|U_n^{(9)}(.,f,a)\right\|_c\le A(a)\omega\left(\frac{1}{n},f\right)_c\qquad(49)$$

$$\left\|U_n^{(10)}(.,f,a)\right\|_c\le A(a)n^a\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(t,f)_c}{n^2t^{3-a}}dt\le A(a)n^a\omega\left(\frac{1}{n},f\right)_c\int_{\frac{\pi}{n}}^{\pi}\frac{nt+1}{n^2t^{3-a}}dt\le A(a)\omega\left(\frac{1}{n},f\right)_c\qquad(50)$$

$$\left\|U_n^{(11)}(.,f,a)\right\|_c\le A(a)\frac{1}{n^{1-a}}\int_{\frac{\pi}{n}}^{\frac{2\pi}{n}}\omega(t,f)_c dt\le A(a)\omega\left(\frac{1}{n},f\right)_c\qquad(51)$$

$$\left\|U_n^{(12)}(.,f,a)\right\|_c\le A(a)n^a\omega\left(\frac{1}{n},f\right)_c\int_{\pi-\frac{2\pi}{n}}^{\pi-\frac{\pi}{n}}\frac{dt}{\left(t+\frac{\pi}{n}\right)^{1-a}}\le A(a)\omega\left(\frac{1}{n},f\right)_c\ .\qquad(52)$$

Accordingly, estimates (45), (48)-(52) give the right to conclude that,

$$\left\|U_n^{(7)}(.,f,a)\right\|_c \le A(a)\omega\left(\frac{1}{n},f\right)_c \quad . \qquad (53)$$

If we take into account the equality (38) and the scheme of estimation of the expression $U_n^{(7)}(.,f,a)$. We find that

$$\left\|U_n^{(i)}(.,f,a)\right\|_c \le A(a)\omega\left(\frac{1}{n},f\right)_c \qquad (i=13,14) \qquad (54)$$

so, from (38), (44), (53) and (54) we find

$$\left\|U_n^{(4)}(.,f,a)\right\|_c \le A(p,a)n^a\omega\left(\frac{1}{n},f\right)_p, \quad ap>1 \qquad (55)$$

$$\left\|U_n^{(4)}(.,f,a)\right\|_c \le A(p)n^{\frac{1}{p}}(\ln n)^{1-1/p}\omega\left(\frac{1}{n},f\right)_p, \quad ap=1 \qquad (56)$$

$$\left\|U_n^{(4)}(.,f,a)\right\|_c \le A(p,a)n^{\frac{1}{p}}\omega\left(\frac{1}{n},f\right)_p, \quad ap<1 \quad . \qquad (57)$$

Accordingly, relations (32)-(34) and (55)-(57) prove the truth of the theorem 1.
The following is also true

Theorem 2 a) Let $a\in\left]0,1\right[$ - is a number. If function $f\in C(T), a, ap>1$ than

$$\left\|t_n^{-a}(f)-\bar{f}_n\right\|_c \le A(p,a)n^a\omega\left(\frac{1}{n},f\right)_p + A(a)g(n,f);$$

b) Let $p\in\left]1,\infty\right[$ - is a number, and function $f\in C(T)$. Than

$$\left\|t_n^{-\frac{1}{p}}(f)-\bar{f}_n\right\|_c \le A(p)\left[n^{\frac{1}{p}}(\ln n)^{1-1/p}\omega\left(\frac{1}{n},f\right)_p + g(n,f)\right].;$$

c) Consider that $p\in\left]0,+\infty\right[$ - is a number and function $f\in C(T)$. If $ap\in\left]0,1\right[$ than

$$\left\|t_n^{-a}(f)-\bar{f}_n\right\|_c \le A(p,a)n^{\frac{1}{p}}\omega\left(\frac{1}{n},f\right)_p + A(a)g(n,f), \quad n\ge 4.$$

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