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# Maximal Subgroups of the Semigroup $B_X(D)$ Defined by Semilattices of the Class $\Sigma_2(X,8)$

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***Abstract.*** *In the given paper we give a full description maximal subgroups of the Complete Semigroups* $B_X(D)$ ***defined by semi lattices of the class*** $\Sigma_2(X,8)$.

***Key words: Semi lattice, Semi group, Binary relation, Idempotent element.*** **AMS Subject Classification. 20M05.**

## I. INTRODUCTION

Let $X$ be an arbitrary nonempty set, $D$ be a $X-$semi lattice of unions, i.e. a nonempty set of subsets of the set $X$ that is closed with respect to the set-theoretic operations of unification of elements from $D$, $f$ be an arbitrary mapping from $X$ into $D$. To each such a mapping $f$ there corresponds a binary relation $\alpha_f$ on the set $X$ that satisfies the condition $\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such $\alpha_f$ $(f: X\to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a $X-$semilattice of unions $D$ (see ([1]).

By $\varnothing$ we denote an empty binary relation or empty subset of the set $X$. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Further let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $T\in D$, $\varnothing\neq D'\subseteq D$ and $t\in\breve{D}=\bigcup_{Y\in D}Y$.

Then by symbols we denote the following sets:

$$y\alpha=\{x\in X \mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y}y\alpha,\ V(D,\alpha)=\{Y\alpha \mid Y\in D\},$$
$$X^*=\{T \mid \varnothing\neq T\subseteq X\},\ D'_t=\{Z'\in D' \mid t\in Z'\},\ Y_T^\alpha=\{x\in X \mid x\alpha=T\}, \quad (1.1)$$
$$D'_T=\{Z'\in D' \mid T\subseteq Z'\},\ \ddot{D}'_T=\{Z'\in D' \mid Z'\subseteq T\},\ l(D',T)=\cup(D'\setminus D'_T).$$

Under symbol $\wedge(D,D_t)$ we mean an exact lower bound of the set $D_t$ in the semi lattice $D$.

**Definition 1.1.**Let $\varepsilon\in B_X(D)$. If $\varepsilon\circ\varepsilon=\varepsilon$ or $\alpha\circ\varepsilon=\alpha$ for any $\alpha\in B_X(D)$, then $\varepsilon$ is called an idempotent element or called right unit of the semigroup $B_X(D)$ respectively (see [1]-[3]).

**Definition1.2.** We say that a complete $X-$semi lattice of unions $D$ is an $XI-$semi lattice of unions if it satisfies the following two conditions:

**a**) $\wedge(D,D_t)\in D$ for any $t\in\breve{D}$;

**b**) $Z=\bigcup_{t\in Z}\wedge(D,D_t)$ for any nonempty element $Z$ of $D$ (see ([1], [2] and [4]).

**Definition1.3.** The one-to-one mapping $\varphi$ between the complete $X-$semi lattices of unions $D'$ and $D''$ is called a complete isomorphism if the condition $\varphi(\cup D_1)=\bigcup_{T'\in D_1}\varphi(T')$ is fulfilled for each nonempty subset $D_1$ of the semi lattice $D'$ (see ([1], [2] and [5]).

**Definition1.4.** We say that an element $Y$ covers an element $Z$ in the semi lattice $D$ if $Y \supset Z$ and there does not exist an element $T \in D$ such that $Y \supset T \supset Z$ (see ([1] and [2]).

**Definition1.5.** We say that a nonempty element $T$ is a no limiting element of the set $D'$ if $T \setminus l(D',T) \neq \varnothing$ and a nonempty element $T$ is a limiting element of the set $D'$ if $T \setminus l(D',T) = \varnothing$ (see ([1] and [2]).

**Theorem1.1.** Let $X$ be a finite set and $D(\alpha)$ be the set of all those elements $T$ of the semi lattice $Q = V(D,\alpha) \setminus \{\varnothing\}$ which are nonlimiting elements of the set $\ddot{Q}_T$. A binary relation $\alpha$ having a quasinormal representation $\alpha = \bigcup_{T \in V(D,\alpha)} (Y_T^\alpha \times T)$ is an idempotent element of this semigroup iff

**a**) $V(D,\alpha)$ is complete $XI-$ senilities of unions;

**b**) $\bigcup_{T' \in \ddot{D}(\alpha)_T} Y_{T'}^\alpha \supseteq T$ for any $T \in D(\alpha)$;

**c**) $Y_T^\alpha \cap T \neq \varnothing$ for any nonlimiting element of the set $\ddot{D}(\alpha)_T$ (see ([1], [2] and [5]).

**Theorem1.2.** Let $D = \{\breve{D}, Z_1, Z_2, ..., Z_{m-1}\}$ be some finite $X-$semilattice of unions and $C(D) = \{P_0, P_1, P_2, ..., P_{m-1}\}$ be the family of sets of pairwise nonintersecting subsets of the set $X$. If $\varphi$ is a mapping of the semilattice $D$ on the family of sets $C(D)$ which satisfies the condition $\varphi(\breve{D}) = P_0$ and $\varphi(Z_i) = P_i$ for any $i = 1, 2, ..., m-1$ and $\hat{D}_Z = D \setminus \{T \in D \mid Z \subseteq T\}$, then the following equalities are valid:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup ... \cup P_{m-1}, \; Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T). \tag{1.2}$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice $D$ are represented in the form (1.2), then among the parameters $P_i$ $(i = 0, 1, 2, ..., m-1)$ there exist such parameters that cannot be empty sets. Such sets $P_i$ $(0 < i \leq m-1)$ are called basis sources, whereas sets $P_j$ $(0 \leq j \leq m-1)$ which can be empty sets too are called completeness sources.

The number the basis sources we denote by the symbol $\delta$.

It is proved that under the mapping $\varphi$ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping $\varphi$ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see ([1]-[3]).

Denote by the symbol $G_X(D,\varepsilon)$ a maximal subgroup of the semigroup $B_X(D)$ whose unit is an idempotent binary relation $\varepsilon$ of the semigroup $B_X(D)$.

**Theorem1.3.** For any idempotent element $\varepsilon \in B_X(D)$, the group $G_X(D,\varepsilon)$ is antiisomorphic to the group of all complete automorphism of the semi lattice $V(D,\varepsilon)$ (see ([1], [2] and [6]).

## II. RESULTS

By the symbol $\Sigma_2(X,8)$ we denote the class of all $X-$semi lattices of unions whose every element is isomorphic to an $X-$semi lattice of form $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, where

$$\begin{array}{llll} Z_6 \subset Z_3 \subset Z_1 \subset \breve{D}, & Z_6 \subset Z_4 \subset Z_1 \subset D, & Z_6 \subset Z_4 \subset Z_2 \subset \breve{D}, & \\ Z_7 \subset Z_4 \subset Z_1 \subset \breve{D}, & Z_7 \subset Z_4 \subset Z_2 \subset \breve{D}, & Z_7 \subset Z_5 \subset Z_2 \subset \breve{D}, & \\ Z_1 \setminus Z_2 \neq \varnothing, & Z_2 \setminus Z_1 \neq \varnothing, & Z_3 \setminus Z_4 \neq \varnothing, & Z_4 \setminus Z_3 \neq \varnothing, \\ Z_3 \setminus Z_5 \neq \varnothing, & Z_5 \setminus Z_3 \neq \varnothing, & Z_4 \setminus Z_5 \neq \varnothing, & Z_5 \setminus Z_4 \neq \varnothing, \\ Z_6 \setminus Z_7 \neq \varnothing, & Z_7 \setminus Z_6 \neq \varnothing, & & \end{array} \tag{2.1}$$

The semi lattice satisfying the conditions $(2.1)$ is shown in Fig. 1. Let $C(D)=\{P_0,P_1,P_2,P_3,P_4,P_5,P_6,P_7\}$ is a family sets, where $P_0,P_1,P_2,P_3,P_4,P_5,P_6,P_7$ are pairwise disjoint subsets of the set $X$ and

$$\varphi=\begin{pmatrix}\breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7\\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7\end{pmatrix}$$

is a mapping of the semi lattice $D$ onto the family sets $C(D)$. Then for the formal equalities of the semilattice $D$ we have a form:

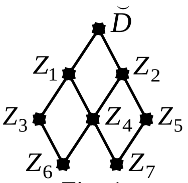

Fig. 1

Diagram of the Semi lattice $D$.

$$\begin{aligned}
\breve{D} &= P_0\cup P_1\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_1 &= P_0\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_2 &= P_0\cup P_1\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_3 &= P_0\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7\\
Z_4 &= P_0\cup P_3\cup P_5\cup P_6\cup P_7\\
Z_5 &= P_0\cup P_1\cup P_3\cup P_4\cup P_6\cup P_7\\
Z_6 &= P_0\cup P_5\cup P_7\\
Z_7 &= P_0\cup P_3\cup P_6
\end{aligned} \qquad (2.2)$$

Here the elements $P_1,P_2,P_3,P_5$ are basis sources, the element $P_0,P_4,P_6,P_7$ are sources of completeness of the semilattice $D$. Therefore $|X|\geq 4$ and $\delta=4$ (see Theorem 1.2).

**Lemma2.1. Let** $D\in\Sigma_2(X,8)$. *Then the following sets exhaust all subsemilattices of the semi lattice* $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$:

**1)** $\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}$, (*see diagram* 1 *of the fig.* 2);

**2)** $\{Z_7,Z_5\},\{Z_7,Z_4\},\{Z_7,Z_2\},\{Z_7,Z_1\},\{Z_7,\breve{D}\},\{Z_6,Z_4\},\{Z_6,Z_3\},$
$\{Z_6,Z_2\},\{Z_6,Z_1\},\{Z_6,\breve{D}\},\{Z_5,Z_2\},\{Z_5,\breve{D}\},\{Z_4,Z_2\},\{Z_4,Z_1\},$
$\{Z_4,\breve{D}\},\{Z_3,Z_1\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}$ (*see diagram* 2 *of the fig.* 2);

**3)** $\{Z_7,Z_5,Z_2\},\{Z_7,Z_5,\breve{D}\},\{Z_7,Z_4,Z_2\},\{Z_7,Z_4,Z_1\},\{Z_7,Z_4,\breve{D}\},\{Z_7,Z_2,\breve{D}\},\{Z_7,Z_1,\breve{D}\},$
$\{Z_6,Z_4,Z_2\},\{Z_6,Z_4,\breve{D}\},\{Z_6,Z_4,Z_1\},\{Z_6,Z_2,\breve{D}\},\{Z_6,Z_3,Z_1\},\{Z_6,Z_3,\breve{D}\},\{Z_6,Z_1,\breve{D}\},$
$\{Z_5,Z_2,\breve{D}\},\{Z_4,Z_2,\breve{D}\},\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_1,\breve{D}\}$ (*see diagram* 3 *of the fig.* 2);

**4)** $\{Z_7,Z_5,Z_2,\breve{D}\},\{Z_7,Z_4,Z_2,\breve{D}\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_1,\breve{D}\},\{Z_6,Z_3,Z_1,\breve{D}\}$
(*see diagram* 4 *of the fig.* 2);

**5)** $\{Z_7,Z_5,Z_4,Z_2\},\{Z_7,Z_5,Z_1,\breve{D}\},\{Z_7,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1\},$
$\{Z_6,Z_3,Z_2,\breve{D}\},\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\}$ (*see diagram* 5 *of the fig.* 2);

**6)** $\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}$ (*see diagram* 6 *of the fig.* 2);

**7)** $\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1,\breve{D}\}$ (*see diagram* 7 *of the fig.* 2);

**8)** $\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 8 *of the fig.* 2);

**9)** $\{Z_7,Z_6,Z_4\},\{Z_7,Z_3,Z_1\},\{Z_6,Z_5,Z_2\},\{Z_5,Z_3,\breve{D}\},\{Z_5,Z_4,Z_2\},$
$\{Z_5,Z_1,\breve{D}\},\{Z_4,Z_3,Z_1\},\{Z_3,Z_2,\breve{D}\},\{Z_2,Z_1,\breve{D}\}$ (*see diagram* 9 *of the fig.* 2);

**10)** $\{Z_7,Z_6,Z_4,Z_2\},\{Z_7,Z_6,Z_4,Z_1\},\{Z_7,Z_6,Z_4,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\},$
$\{Z_6,Z_5,Z_2,\breve{D}\},\{Z_5,Z_4,Z_2,\breve{D}\},\{Z_4,Z_3,Z_1,\breve{D}\}$ (*see diagram* 10 *of the fig.* 2);

**11)** $\{Z_7,Z_6,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_4,Z_1,\breve{D}\}$ (*see diagram* 11 *of the fig.* 2);

**12)** $\{Z_7,Z_6,Z_4,Z_2,Z_1,\breve{D}\}$ (*see diagram* 12 *of the fig.* 2);

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**13)** $\{Z_7,Z_6,Z_5,Z_4,Z_2\},\{Z_7,Z_6,Z_4,Z_3,Z_1\},\{Z_7,Z_5,Z_3,Z_1,\breve{D}\},\{Z_6,Z_5,Z_3,Z_2,\breve{D}\},$
$\{Z_7,Z_3,Z_2,Z_1,\breve{D}\},\{Z_4,Z_3,Z_2,Z_1,\breve{D}\},\{Z_6,Z_5,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2,Z_1,\breve{D}\}$
(*see diagram* 13 *of the fig.* 2);

**14)** $\{Z_7,Z_6,Z_5,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_4,Z_3,Z_1,\breve{D}\}$ (*see diagram* 14 *of the fig.* 2);

**15)** $\{Z_7,Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\},\{Z_7,Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 15 *of the fig.* 2);

**16)** $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 16 *of the fig.* 2);

**17)** $\{Z_7,Z_4,Z_3,Z_1\},\{Z_6,Z_5,Z_4,Z_2\},\{Z_5,Z_3,Z_2,\breve{D}\},\{Z_5,Z_2,Z_1,\breve{D}\},\{Z_5,Z_3,Z_1,\breve{D}\},\{Z_3,Z_2,Z_1,\breve{D}\}$
(*see diagram* 17 *of the fig.* 2);

**18)** $\{Z_5,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 18 *of the fig.* 2);

**19)** $\{Z_7,Z_5,Z_2,Z_1,\breve{D}\},\{Z_6,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 19 *of the fig.* 2);

**20)** $\{Z_7,Z_4,Z_3,Z_1,\breve{D}\},\{Z_6,Z_5,Z_4,Z_2,\breve{D}\}$ (*see diagram* 20 *of the fig.* 2);

**21)** $\{Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 21 *of the fig.* 2);

**22)** $\{Z_7,Z_5,Z_3,Z_2,Z_1,\breve{D}\},\{Z_6,Z_5,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 22 *of the fig.* 2);

**23)** $\{Z_7,Z_4,Z_3,Z_2,Z_1,\breve{D}\},\{Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\}$ (*see diagram* 23 *of the fig.* 2);

**24)** $\{Z_7,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\},\{Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 24 *of the fig.* 2).

*Proof.* As is easily seen, the sets $\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}$ are subsemilattices of the senilities $D$.

The number all subsets of the semi lattice $D$, every set of which contains two elements, is equal to $C_8^2 = 28$. In this case $X$ - subsemilattices of the semi lattice $D$ are the following sets:

$$\{Z_7,Z_5\},\{Z_7,Z_4\},\{Z_7,Z_2\},\{Z_7,Z_1\},\{Z_7,\breve{D}\},\{Z_6,Z_4\},\{Z_6,Z_3\},\{Z_6,Z_2\},\{Z_6,Z_1\},\{Z_6,\breve{D}\},$$
$$\{Z_5,Z_2\},\{Z_5,\breve{D}\},\{Z_4,Z_2\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,Z_1\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}.$$

Remainder nine subsets of the semi lattice $D$, whose every element contains two elements is not an $X$ - subsemilattice.

The number all subsets of the semi lattice $D$, every set of which contains three elements, is equal to $C_8^3 = 56$. In this case $X$ - subsemilattices of the semi lattice $D$ are the following sets:

$$\{Z_7,Z_5,Z_2\},\{Z_7,Z_5,\breve{D}\},\{Z_7,Z_4,Z_2\},\{Z_7,Z_4,Z_1\},\{Z_7,Z_4,\breve{D}\},\{Z_7,Z_2,\breve{D}\},\{Z_7,Z_1,\breve{D}\},$$
$$\{Z_6,Z_4,Z_2\},\{Z_6,Z_4,\breve{D}\},\{Z_6,Z_4,Z_1\},\{Z_6,Z_2,\breve{D}\},\{Z_6,Z_3,Z_1\},\{Z_6,Z_3,\breve{D}\},\{Z_6,Z_1,\breve{D}\},$$
$$\{Z_5,Z_2,\breve{D}\},\{Z_4,Z_2,\breve{D}\},\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_1,\breve{D}\},\{Z_7,Z_6,Z_4\},\{Z_7,Z_3,Z_1\},\{Z_6,Z_5,Z_2\},$$
$$\{Z_5,Z_3,\breve{D}\},\{Z_5,Z_4,Z_2\},\{Z_5,Z_1,\breve{D}\},\{Z_4,Z_3,Z_1\},\{Z_3,Z_2,\breve{D}\},\ \{Z_2,Z_1,\breve{D}\}.$$

Remainder twenty-nine subsets of the semi lattice $D$, whose every element contains three elements is not an $X$ - subsemilattice.

The number all subsets of the semi lattice $D$, every set of which contains four elements, is equal to $C_8^4 = 70$. In this case $X$ - subsemilattices of the semi lattice $D$ are the following sets:

$$\{Z_7,Z_5,Z_2,\breve{D}\},\{Z_7,Z_4,Z_2,\breve{D}\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_1,\breve{D}\},\{Z_6,Z_3,Z_1,\breve{D}\}$$
$$\{Z_7,Z_5,Z_4,Z_2\},\{Z_7,Z_5,Z_1,\breve{D}\},\{Z_7,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1\},\{Z_6,Z_3,Z_2,\breve{D}\},\{Z_6,Z_2,Z_1,\breve{D}\},$$
$$\{Z_4,Z_2,Z_1,\breve{D}\},\{Z_7,Z_6,Z_4,Z_2\},\{Z_7,Z_6,Z_4,Z_1\},\{Z_7,Z_6,Z_4,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\},\{Z_6,Z_5,Z_2,\breve{D}\},$$
$$\{Z_5,Z_4,Z_2,\breve{D}\},\{Z_4,Z_3,Z_1,\breve{D}\},\{Z_7,Z_4,Z_3,Z_1\},\{Z_6,Z_5,Z_4,Z_2\},\{Z_5,Z_3,Z_2,\breve{D}\},\{Z_5,Z_2,Z_1,\breve{D}\},$$
$$\{Z_5,Z_3,Z_1,\breve{D}\},\{Z_3,Z_2,Z_1,\breve{D}\}.$$

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Remainder forty-four subsets of the semi lattice $D$, whose every element contains four elements is not an $X$ - subsemilattice.

The number all subsets of the semi lattice $D$, every set of which contains five elements, is equal to $C_8^5 = 56$. In this case $X$ - subsemilattices of the semi lattice $D$ are the following sets:

$$\{Z_7, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_3, Z_1\}, \{Z_7, Z_5, Z_3, Z_1, \breve{D}\},$$
$$\{Z_6, Z_5, Z_3, Z_2, \breve{D}\}, \{Z_7, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_2, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_2, Z_1, \breve{D}\},$$
$$\{Z_5, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_2, \breve{D}\},$$
$$\{Z_7, Z_5, Z_4, Z_2, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_1, \breve{D}\}.$$

Remainder thirty-seven subsets of the semi lattice $D$, whose every element contains five elements is not an $X$ - subsemilattice.

The number all subsets of the semi lattice $D$, every set of which contains six elements, is equal to $C_8^6 = 28$. In this case $X$ - subsemilattices of the semi lattice $D$ are the following sets:

$$\{Z_7, Z_5, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_6, Z_5, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_5, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}.$$

Remainder eight-teen subset of the semi lattice $D$, whose every element contains six elements is not an $X$ - subsemilattice.

The number all subsets of the semi lattice $D$, every set of which contains seven elements, is equal to $C_8^7 = 8$. In this case $X$ - subsemilattices of the semi lattice $D$ are the following sets:

$$\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}.$$

Remainder four subsets of the semi lattice $D$, whose every element contains six elements is not an $X$ - subsemilattice.

The Lemma is proved.

From the proven lemma it follows that diagrams shown in fig. 2, exhaust all diagrams of subsemilattices of the semi lattice $D$.

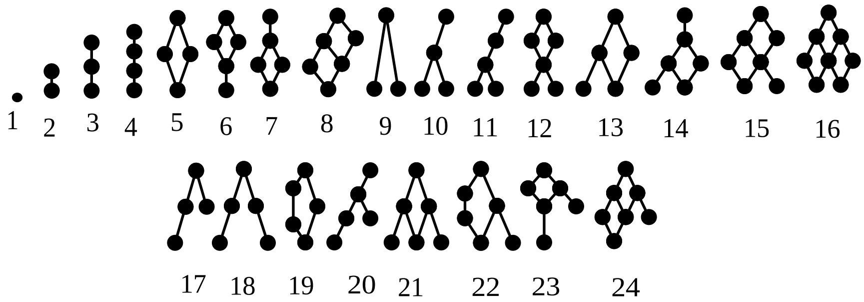


Fig. 2

**Diagrams of all subsemilattices of the semi lattice $D$.**

**Lemma2.2.** *Let* $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8)$. *Then any subsemilattice of the semi lattice* $D$ *having diagram* $17-24$ *of the figure 2 are never* $XI-$ *semi lattice.*

*Proof.* Note that the subsemilattices of the semi lattice $D$ having diagram of form 17-24 are never $XI-$ semi lattices. As an example consider a semi lattice such is defined by the diagram of form 24 show in figure 2. let $D' = \{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ and $C(D') = \{P_0, P_1, P_2, P_3, P_4, P_5, P_6\}$ is a family sets, where

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$P_0, P_1, P_2, P_3, P_4, P_5, P_6$ are pairwise disjoint subsets of the set $X$ and $\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{pmatrix}$ is a mapping of the semi lattice $D'$ onto the family sets $C(D')$. Then for the formal equalities of the semi lattice $D'$ (see figure 3) we have a form:

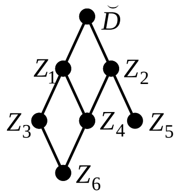

Fig. 3

$$\begin{aligned} \breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \\ Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \\ Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \\ Z_3 &= P_0 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \\ Z_4 &= P_0 \cup P_3 \cup P_5 \cup P_6 \\ Z_5 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_6 \\ Z_6 &= P_0 \cup P_5 \end{aligned} \qquad (2.3)$$

Diagram of the Semi lattice $D'$.

Here the elements $P_1, P_2, P_3, P_5$ are basis sources, while the element $P_0, P_4, P_6$ are sources of completeness of the semi lattice $D'$. Therefore $|X| \geq 3$ and $\delta = 4$ (see Theorem 1.2). Then from the formal equalities is follows that

$$D'_t = \begin{cases} D', \ if \ \ t \in P_0, \\ \{Z_5, Z_2, \breve{D}\}, \ if \ \ t \in P_1, \\ \{Z_3, Z_1, \breve{D}\}, \ if \ \ t \in P_2, \\ \{Z_5, Z_4, Z_2, Z_1, \breve{D}\}, \ if \ \ t \in P_3, \\ \{Z_5, Z_3, Z_2, Z_1, \breve{D}\}, \ if \ \ t \in P_4, \\ \{Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \ if \ \ t \in P_5, \\ \{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \ if \ \ t \in P_6, \end{cases} \qquad \wedge(D', D'_t) = \begin{cases} Z_5, \ if \ \ t \in P_1, \\ Z_3, \ if \ \ t \in P_2, \\ Z_6, \ if \ \ t \in P_5. \end{cases}$$

We have $D'^{\wedge} = \{\wedge(D', D'_t) | t \in \breve{D}\} = \{Z_6, Z_5, Z_3\}$ and $\wedge(D', D'_t) \notin D'$ if $t \in P_0 \cup P_3 \cup P_4 \cup P_6 \supseteq P_3 \neq \varnothing$ since $P_3$ is basis source. Hence Definition 1.2 implies that a subsemilattice of the semilattice $D'$ that has diagram 24 is never $XI-$ semi lattice.

In the same manner it can be proved that any subsemilattice of the semilattice $D$ having diagrams 17-24 are never an $XI-$ semi lattice.

The Lemma is proved.

**Lemma2.3.** *Let* $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8)$ *and* $Z_7 \cap Z_6 \neq \varnothing$. *Then the following sets exhibit all* $XI-$ *subsemilattices of the given semilattice* $D$ *:*

**1)** $\{\breve{D}\}, \{Z_1\}, \{Z_2\}, \{Z_3\}, \{Z_4\}, \{Z_5\}, \{Z_6\}, \{Z_7\}$ (*see diagram* 1 *of the fig.* 4);

**2)** $\{Z_7, Z_5\}, \{Z_7, Z_4\}, \{Z_7, Z_2\}, \{Z_7, Z_1\}, \{Z_7, \breve{D}\}, \{Z_6, Z_4\}, \{Z_6, Z_3\}, \{Z_6, Z_2\}, \{Z_6, Z_1\}, \{Z_6, \breve{D}\},$
$\{Z_5, Z_2\}, \{Z_5, \breve{D}\}, \{Z_4, Z_2\}, \{Z_4, Z_1\}, \{Z_4, \breve{D}\}, \{Z_3, Z_1\}, \{Z_3, \breve{D}\}, \{Z_2, \breve{D}\}, \{Z_1, \breve{D}\}$
(*see diagram* 2 *of the fig.* 4);

**3)** $\{Z_7, Z_5, Z_2\}, \{Z_7, Z_5, \breve{D}\}, \{Z_7, Z_4, Z_2\}, \{Z_7, Z_4, Z_1\}, \{Z_7, Z_4, \breve{D}\}, \{Z_7, Z_2, \breve{D}\}, \{Z_7, Z_1, \breve{D}\},$
$\{Z_6, Z_4, Z_2\}, \{Z_6, Z_4, \breve{D}\}, \{Z_6, Z_4, Z_1\}, \{Z_6, Z_2, \breve{D}\}, \{Z_6, Z_3, Z_1\}, \{Z_6, Z_3, \breve{D}\}, \{Z_6, Z_1, \breve{D}\},$
$\{Z_5, Z_2, \breve{D}\}, \{Z_4, Z_2, \breve{D}\}, \{Z_4, Z_1, \breve{D}\}, \{Z_3, Z_1, \breve{D}\}$ (*see diagram* 3 *of the fig.* 4);

**4)** $\{Z_7, Z_5, Z_2, \breve{D}\}, \{Z_7, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_4, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_2, \breve{D}\}, \{Z_6, Z_4, Z_1, \breve{D}\}, \{Z_6, Z_3, Z_1, \breve{D}\}$
(*see diagram* 4 *of the fig.* 4);

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**5)** $\{Z_7,Z_5,Z_4,Z_2\},\{Z_7,Z_5,Z_1,\breve{D}\},\{Z_7,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1\},\{Z_6,Z_3,Z_2,\breve{D}\},$
$\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\}$ (*see diagram* 5 *of the fig*. 4);

**6)** $\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}$ (*see diagram* 6 *of the fig*. 4);

**7)** $\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1,\breve{D}\}$ (*see diagram* 7 *of the fig*. 4);

**8)** $\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ (*see diagram* 8 *of the fig*. 4).

*Proof.* The statements $1)-4)$ immediately follows from the Theorems 11.6.1 in [1], 11.6.1 in [2], the statements $5)-7)$ immediately follows from the Theorems 11.6.3 in [1], 11.6.3 in [2] and the statement $8)$ immediately follows from the Theorems 11.7.2 in [1].
The Lemma is proved.

**Lemma2.4.** *Let* $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ *and* $Z_7\cap Z_6=\varnothing,\ Z_7\cap Z_3\neq\varnothing,\ Z_6\cap Z_5\neq\varnothing,$ *than semi lattices from the lemma 2.3 and the following sets exhibit all XI -subsemilattices of the given semilattice* $D$ *:*

**9)** $\{Z_7,Z_6,Z_4\}$, (*see diagram* 9 *of the fig*. 4),
**10)** $\{Z_7,Z_6,Z_4,Z_2\}$, $\{Z_7,Z_6,Z_4,Z_1\}$, $\{Z_7,Z_6,Z_4,\breve{D}\}$ (*see diagram* 10 *of the fig*. 4),
**11)** $\{Z_7,Z_6,Z_4,Z_2,\breve{D}\}$, $\{Z_7,Z_6,Z_4,Z_1,\breve{D}\}$ (*see diagram* 11 *of the fig*. 4),
**12)** $\{Z_7,Z_6,Z_4,Z_2,Z_1,\breve{D}\}$, (*see diagram* 12 *of the fig*. 4).

*Proof.* The statements $9)-11)$ immediately follows from the Theorems 11.6.2 in [1], 11.6.2 in [2] and the statement $12)$ immediately follows from the Theorems 11.6.5 in [1], 11.6.5 in [2] .
The Lemma is proved.

**Lemma2.5.** *Let* $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ *and* $Z_7\cap Z_6=\varnothing,\ Z_7\cap Z_3=\varnothing,\ Z_6\cap Z_5\neq\varnothing,$ *than semi lattices from the lemma 2.4 and the following sets exhibit all* $XI-$*subsemilattices of the given semilattice* $D$ *:*

**9)** $\{Z_7,Z_3,Z_1\}$, (*see diagram* 9 *of the fig*. 4),
**10)** $\{Z_7,Z_3,Z_1,\breve{D}\}$, (*see diagram* 10 *of the fig*. 4),
**13)** $\{Z_7,Z_6,Z_4,Z_3,Z_1\}$, (*see diagram* 13 *of the fig*. 4),
**14)** $\{Z_7,Z_6,Z_4,Z_3,Z_1,\breve{D}\}$, (*see diagram* 14 *of the fig*. 4),
**15)** $\{Z_7,Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, (*see diagram* 15 *of the fig*. 4).

*Proof.* The statements $9)$, $10)$ immediately follows from the Theorems 11.6.2 in [1], 11.6.2 in [2], the statements $13)$, $14)$ immediately follows from the Theorems 11.6.4 in [1], 11.6.4 in [2] and the statement $15)$ immediately follows from the Theorems 13.10.1 in [1].
The Lemma is proved.

**Lemma2.6.** *Let* $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ $Z_7\cap Z_6=\varnothing,\ Z_6\cap Z_5=\varnothing,\ Z_7\cap Z_3\neq\varnothing,$ *than semi lattices from the lemma 2.4 and the following sets exhibit all* $XI-$*subsemilattices of the given semilattice* $D$ *:*

**9)** $\{Z_6,Z_5,Z_2\}$, (*see diagram* 9 *of the fig*. 4),
**10)** $\{Z_6,Z_5,Z_2,\breve{D}\}$, (*see diagram* 10 *of the fig*. 4),
**13)** $\{Z_7,Z_6,Z_5,Z_4,Z_2\}$, (*see diagram* 13 *of the fig*. 4),
**14)** $\{Z_7,Z_6,Z_5,Z_4,Z_2,\breve{D}\}$, (*see diagram* 14 *of the fig*. 4),
**15)** $\{Z_7,Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\}$, (*see diagram* 15 *of the fig*. 4).

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*Proof.* The statements $9)$, $10)$ immediately follows from the Theorems 11.6.2 in [1], 11.6.2 in [2], the statements $13)$, $14)$ immediately follows from the Theorems 11.6.4 in [1], 11.6.4 in [2] and the statement $15)$ immediately follows from the Theorems 13.10.1 in [1].
The Lemma is proved.

**Lemma2.7.** *Let* $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ *and* $Z_7\cap Z_6=\varnothing$, $Z_7\cap Z_3=\varnothing$, $Z_6\cap Z_5=\varnothing$, $Z_5\cap Z_3\neq\varnothing$, *than semilattices from the lemma 2.5 and 2.6 are* $XI-$*subsemilattices of the given semilattice* $D$.

*Proof.* This Lemma immediately follows from the Lemmas 2.5 and 2.6.
The Lemma is proved.

**Lemma2.8.** *Let* $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ *and* $Z_5\cap Z_3=\varnothing$, *than all semilattisec from the lemma 2.7 and the following sets exhibit all* $XI-$*subsemilattices of the given semilattice* $D$:

**9)** $\{Z_5,Z_3,\breve{D}\}$, (*see diagram* 9 *of the fig*. 4),
**13)** $\{Z_6,Z_5,Z_3,Z_2,\breve{D}\}$, $\{Z_7,Z_5,Z_3,Z_1,\breve{D}\}$ (*see diagram* 13 *of the fig*. 4),
**16)** $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, (*see diagram* 16 *of the fig*. 4).

*Proof.* The statement $9)$ immediately follows from the Theorems 11.6.2 in [1], 11.6.2 in [2], the statement $13)$ immediately follows from the Theorems 11.6.4 in [1], 11.6.4 in [2].
Now we will proof the statement $16)$. Indeed, let $t\in\breve{D}$, $D_t=\{Z\in D\,|\,t\in Z\}$ and $\wedge(D,D_t)$ is the exact lower bound of the set $D_t$ in $D$. Then of the formal equalities $(2.2)$ follows, that

$$D_t=\begin{cases}D, \text{ if } t\in P_0,\\ \{Z_5,Z_2,\breve{D}\}, \text{ if } t\in P_1,\\ \{Z_3,Z_1,\breve{D}\}, \text{ if } t\in P_2,\\ \{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\}, \text{ if } t\in P_3,\\ \{Z_5,Z_3,Z_2,Z_1,\breve{D}\}, \text{ if } t\in P_4,\\ \{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}, \text{ if } t\in P_5,\\ \{Z_7,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}, \text{ if } t\in P_6,\\ \{Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}, \text{ if } t\in P_7,\end{cases}\qquad \wedge(D,D_t)=\begin{cases}Z_5, \text{ if } t\in P_1,\\ Z_3, \text{ if } t\in P_2,\\ Z_7, \text{ if } t\in P_3,\\ Z_6, \text{ if } t\in P_5.\end{cases}$$

We have $D^\wedge=\{\wedge(D,D_t)\,|\,t\in\breve{D}\}=\{Z_7,Z_6,Z_5,Z_3\}$ and $\wedge(D,D_t)\notin D$ if $t\in P_0\cup P_4\cup P_6\cup P_7$. So, from the Definition 1.2 follows that semilattice $D$ is not $XI-$semilattice.
If $P_0=P_4=P_6=P_7=\varnothing$ since they are completeness sources, then $\wedge(D,D_t)\in D$ for all $t\in\breve{D}$ and $Z_4=Z_7\cup Z_6$, $Z_1=Z_7\cup Z_3$, $Z_2=Z_6\cup Z_5$. Of the last conditions and from the Definition 1.2 follows that the semilattice $D$ is $XI-$semilattice. Of the equality $P_0=P_4=P_6=P_7=\varnothing$ follows that

$$Z_5\cap Z_3=(P_0\cup P_1\cup P_3\cup P_4\cup P_6\cup P_7)\cap(P_0\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7)=P_0\cup P_4\cup P_6\cup P_7=\varnothing.$$

Of the other hand, if $Z_5\cap Z_3=\varnothing$ then by formal equalities follows that $P_0=P_4=P_6=P_7=\varnothing$. Therefore, semilattice $D$ is $XI-$semilattice.
The Lemma is proved.

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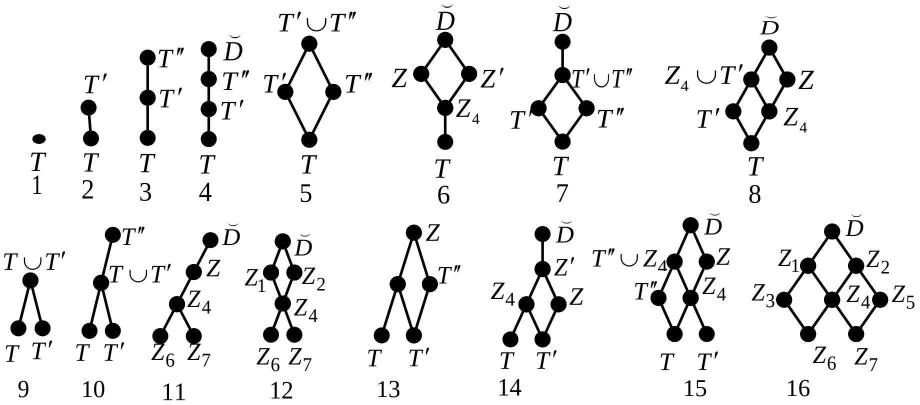

Fig.4

**Diagrams of all $XI$ – subsemilattice of the semilattice $D$.**

**Theorem2.1.** *Let $D \in \Sigma_2(X,8)$, $Z_7 \cap Z_6 \neq \varnothing$ and $\alpha \in B_X(D)$. Binary relation $\alpha$ is an idempotent relation of the semmigroup $B_X(D)$ iff binary relation $\alpha$ satisfies only one conditions of the following conditions:*

**1)** $\alpha = X \times T$, where $T \in D$;

**2)** $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$, where $T, T' \in D$, $T \subset T'$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \supseteq T$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

**3)** $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$, where $T, T', T'' \in D$, $T \subset T' \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \supseteq T$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

**4)** $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_0^\alpha \times \breve{D})$, where $T, T', T'' \in D$, $T \subset T' \subset T'' \subset \breve{D}$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \supseteq T$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

**5)** $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T''))$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$,
$T'' \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

**6)** $\alpha = (Y_T^\alpha \times T) \cup (Y_4^\alpha \times Z_4) \cup (Y_Z^\alpha \times Z) \cup (Y_{Z'}^\alpha \times Z') \cup (Y_0^\alpha \times \breve{D})$, where $T \in \{Z_7, Z_6\}$, $Z, Z' \in \{Z_2, Z_1\}$, $Z \neq Z'$ $Z \setminus Z' \neq \varnothing$, $Z' \setminus Z \neq \varnothing$, $Y_T^\alpha, Y_4^\alpha, Y_Z^\alpha, Y_{Z'}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \supseteq T$, $Y_T^\alpha \cup Y_4^\alpha \supseteq Z_4$, $Y_T^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq Z$, $Y_T^\alpha \cup Y_4^\alpha \cup Y_{Z'}^\alpha \supseteq Z'$, $Y_4^\alpha \cap Z_4 \neq \varnothing$, $Y_Z^\alpha \cap Z \neq \varnothing$, $Y_{Z'}^\alpha \cap Z' \neq \varnothing$;

**7)** $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T'')) \cup (Y_0^\alpha \times \breve{D})$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

**8)** $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_4^\alpha \times Z_4) \cup (Y_{T' \cup Z_4}^\alpha \times (T' \cup Z_4)) \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \breve{D})$, where $T \in \{Z_7, Z_6\}$, $T' \in \{Z_5, Z_3\}$, $Z_4 \cup T'$, $Z \in \{Z_2, Z_1\}$, $Z_4 \cup T' \neq Z$, $T' \setminus Z_4 \neq \varnothing$, $Z_4 \setminus T' \neq \varnothing$, $(Z_4 \cup T') \setminus Z \neq \varnothing$,

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$Z\setminus(Z_4\cup T')\neq\varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_4^\alpha, Y_Z^\alpha \notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_Z^\alpha\cap Z\neq\varnothing$.

*Proof.* The statements $1)-4)$ immediately follows from the Corollary 13.1.1 in [1], 13.1.1 in [2], the statements $5)-7)$ immediately follows from the Corollary 13.3.1 in [1], 13.3.1 in [2], the statement $8)$ immediately follows from the Theorem 13.7.2 in [1], 13.7.2 in [2].

The Theorem is proved.

**Theorem2.2.** *Let $D\in\Sigma_2(X,8)$, $Z_7\cap Z_6=\varnothing$, $Z_7\cap Z_3\neq\varnothing$, $Z_6\cap Z_5\neq\varnothing$ and $\alpha\in B_X(D)$. Binary relation $\alpha$ is an idempotent relation of the semmigroup $B_X(D)$ iff binary relation $\alpha$ satisfies only one conditions of the Theorem 2.1 and only one following conditions:*

**9)** $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T\cup T'}^\alpha\times(T\cup T'))$, where $T,T'\in D$, $T\setminus T'\neq\varnothing$, $T'\setminus T\neq\varnothing$, $Y_T^\alpha, Y_{T'}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\supseteq T'$;

**10)** $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T\cup T'}^\alpha\times(T\cup T'))\cup(Y_{T''}^\alpha\times T'')$, where $T,T',T''\in D$, $T\setminus T'\neq\varnothing$, $T'\setminus T\neq\varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\supseteq T'$, $Y_{T''}^\alpha\cap T''\neq\varnothing$;

**11)** $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_6^\alpha\times Z_6)\cup(Y_4^\alpha\times Z_4)\cup(Y_Z^\alpha\times Z)\cup(Y_0^\alpha\times\breve{D})$, where $Z\in\{Z_2,Z_1\}$, $Y_7^\alpha, Y_6^\alpha, Y_Z^\alpha, Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z$, $Y_Z^\alpha\cap Z\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

**12)** $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_6^\alpha\times Z_6)\cup(Y_4^\alpha\times Z_4)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_2^\alpha, Y_1^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_1^\alpha\supseteq Z_1$, $Y_2^\alpha\cap Z_2\neq\varnothing$, $Y_1^\alpha\cap Z_1\neq\varnothing$.

*Proof. T*he statements $9)-11)$ immediately follows from the Corollary 13.2.1 in [1], 13.2.1 in [2], the statement $12)$ immediately follows from the Corollary 13.5.1 in [1], 13.5 in [2].

The Theorem is proved.

**Theorem2.3.** *Let $D\in\Sigma_2(X,8)$, $Z_7\cap Z_6=\varnothing$, $Z_7\cap Z_3=\varnothing$, $Z_6\cap Z_5\neq\varnothing$ and $\alpha\in B_X(D)$. Binary relation $\alpha$ is an idempotent relation of the semmigroup $B_X(D)$ iff binary relation $\alpha$ satisfies only one conditions of the Theorem 2.2 and only one following conditions:*

**13)** $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_6^\alpha\times Z_6)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_1^\alpha\times Z_1)$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_6^\alpha\supseteq Z_6$, $Y_6^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_3^\alpha\cap Z_3\neq\varnothing$;

**14)** $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_6^\alpha\times Z_6)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha, Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_6^\alpha\supseteq Z_6$, $Y_6^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

**15)** $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_6^\alpha\times Z_6)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha, Y_2^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_6^\alpha\supseteq Z_6$, $Y_6^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_2^\alpha\cap Z_2\neq\varnothing$.

*Proof.* The statements $13)$, $14)$ immediately follows from the Corollary 13.4.1 in [1], 13.4.1 in [2], the statement $15)$ immediately follows from the Theorem 13.10.2 in [1].

The Theorem is proved.

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**Theorem2.4.** *Let* $D \in \Sigma_2(X,8)$, $Z_7 \cap Z_6 = \varnothing$, $Z_6 \cap Z_5 = \varnothing$, $Z_7 \cap Z_3 \neq \varnothing$ *and* $\alpha \in B_X(D)$. *Binary relation* $\alpha$ *is an idempotent relation of the semmigroup* $B_X(D)$ *iff binary relation* $\alpha$ *satisfies only one conditions of the Theorem 2.2 and only one following conditions:*

**13)** $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2)$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq Z_7$, $Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_5^\alpha \cap Z_5 \neq \varnothing$;

**14)** $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D})$, *where* $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq Z_7$, $Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_5 \cap Z_5 \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

**15)** $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, *where* $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_1^\alpha \notin \{\varnothing\}$ and satisfies theconditions: $Y_7^\alpha \supseteq Z_7$, $Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_1^\alpha \supseteq Z_1$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_1^\alpha \cap Z_1 \neq \varnothing$.

*Proof.* The statements 13), 14) immediately follows from the Corollary 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.2 in [1].

Theorem is proved.

**Theorem2.5.** *Let* $D \in \Sigma_2(X,8)$, $Z_7 \cap Z_6 = \varnothing$, $Z_6 \cap Z_5 = \varnothing$, $Z_7 \cap Z_3 = \varnothing$, $Z_5 \cap Z_3 \neq \varnothing$ *and* $\alpha \in B_X(D)$. *Binary relation* $\alpha$ *is an idempotent relation of the semmigroup* $B_X(D)$ *iff binary relation* $\alpha$ *satisfies only one conditions of the Theorem 2.3 and only one conditions of the Theorem 2.4.*

*Proof.* This Theorem immediately follows from the Theorems 2.3 and 2.4.

The Theorem is proved.

**Theorem2.6.** *Let* $D \in \Sigma_2(X,8)$, $Z_5 \cap Z_3 = \varnothing$ *and* $\alpha \in B_X(D)$. *Binary relation* $\alpha$ *is an idempotent relation of the semmigroup* $B_X(D)$ *iff binary relation* $\alpha$ *satisfies only one conditions of the Theorem 2.5 and only one following condition:*

**13)** $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'') \cup (Y_Z \times Z)$, where $T, T', T'', Z \in D$, $(T \cup T') \subset Z$, $T' \subset T'' \subset Z$, $(T \cup T') \setminus T'' \neq \varnothing$, $T'' \setminus (T \cup T') \neq \varnothing$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \supseteq T$, $Y_{T'}^\alpha \supseteq T'$, $Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

**16)** $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_3^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq Z_7$, $Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_6^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$.

*Proof.* The statement 13) immediately follows from the Corollary 13.4.1 in [1], 13.4.1 in [2]. Now we will proof the statement 3). It is easy to see, that the set $D(\alpha) = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ is a generating set of the semilattice $D$. Then the following equalities are hold:

$$\ddot{D}(\alpha)_{Z_7} = \{Z_7\},\ \ddot{D}(\alpha)_{Z_6} = \{Z_6\},\ \ddot{D}(\alpha)_{Z_5} = \{Z_7, Z_5\},\ \ddot{D}(\alpha)_{Z_4} = \{Z_7, Z_6, Z_4\},$$
$$\ddot{D}(\alpha)_{Z_3} = \{Z_6, Z_3\}, \ddot{D}(\alpha)_{Z_2} = \{Z_7, Z_6, Z_5, Z_4, Z_2\},\ \ddot{D}(\alpha)_{Z_1} = \{Z_7, Z_6, Z_4, Z_3, Z_1\}.$$

By statement $b)$ of the Theorem 1.1 follows that the following conditions are true:

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$$Y_7^\alpha \supseteq Z_7,\ Y_6^\alpha \supseteq Z_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5,\ Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq Z_4,\ Y_6^\alpha \cup Y_3^\alpha \supseteq Z_3,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2,\ Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha \supseteq Z_1;$$

$$Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq Z_7 \cup Z_6 \cup Y_4^\alpha = Z_4 \cup Y_4^\alpha \supseteq Z_4,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha = \left(Y_7^\alpha \cup Y_5^\alpha\right) \cup \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha\right) \cup Y_2^\alpha \supseteq$$
$$\supseteq Z_5 \cup Z_4 \cup Y_2^\alpha = Z_2 \cup Y_2^\alpha \supseteq Z_2,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha = \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha\right) \cup \left(Y_6^\alpha \cup Y_3^\alpha\right) \cup Y_1^\alpha \supseteq$$
$$\supseteq Z_4 \cup Z_3 \cup Y_1^\alpha = Z_1 \cup Y_1^\alpha \supseteq Z_1,$$

i.e., the inclusions

$$Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq Z_4,\ Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2,\ Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha \supseteq Z_1$$

are always hold. Further, it is to see, that the following conditions are true:

$$l\left(\ddot{D}_{Z_7}, Z_7\right) = \cup\left(\ddot{D}_{Z_7} \setminus \{Z_7\}\right) = \varnothing,\ Z_7 \setminus l\left(\ddot{D}_{Z_7}, Z_7\right) = Z_7 \setminus \varnothing \neq \varnothing;$$
$$l\left(\ddot{D}_{Z_6}, Z_6\right) = \cup\left(\ddot{D}_{Z_6} \setminus \{Z_6\}\right) = \varnothing,\ Z_6 \setminus l\left(\ddot{D}_{Z_6}, Z_6\right) = Z_6 \setminus \varnothing \neq \varnothing;$$
$$l\left(\ddot{D}_{Z_5}, Z_5\right) = \cup\left(\ddot{D}_{Z_5} \setminus \{Z_5\}\right) = Z_7,\ Z_5 \setminus l\left(\ddot{D}_{Z_5}, Z_5\right) = Z_5 \setminus Z_7 \neq \varnothing;$$
$$l\left(\ddot{D}_{Z_3}, Z_3\right) = \cup\left(\ddot{D}_{Z_3} \setminus \{Z_3\}\right) = Z_6,\ Z_3 \setminus l\left(\ddot{D}_{Z_3}, Z_3\right) = Z_3 \setminus Z_6 \neq \varnothing;$$

$$l\left(\ddot{D}_{Z_4}, Z_4\right) = \cup\left(\ddot{D}_{Z_4} \setminus \{Z_4\}\right) = Z_4,\ Z_4 \setminus l\left(\ddot{D}_{Z_4}, Z_4\right) = Z_4 \setminus Z_4 = \varnothing;$$
$$l\left(\ddot{D}_{Z_2}, Z_2\right) = \cup\left(\ddot{D}_{Z_2} \setminus \{Z_2\}\right) = Z_2,\ Z_2 \setminus l\left(\ddot{D}_{Z_2}, Z_2\right) = Z_2 \setminus Z_2 = \varnothing;$$
$$l\left(\ddot{D}_{Z_1}, Z_1\right) = \cup\left(\ddot{D}_{Z_1} \setminus \{Z_1\}\right) = Z_1,\ Z_1 \setminus l\left(\ddot{D}_{Z_1}, Z_1\right) = Z_1 \setminus Z_1 = \varnothing.$$

We have the elements $Z_7, Z_6, Z_5, Z_3$ are nonlimiting elements of the sets $\ddot{D}(\alpha)_{Z_7}$, $\ddot{D}(\alpha)_{Z_6}$, $\ddot{D}(\alpha)_{Z_5}$ and $\ddot{D}(\alpha)_{Z_3}$ respectively. By statement $c)$ of the Theorem 1.1 it follows, that the conditions $Y_7^\alpha \cap Z_7 \neq \varnothing$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_5^\alpha \cap Z_5 \neq \varnothing$ and $Y_3^\alpha \cap Z_3 \neq \varnothing$ are hold. Since $Z_7 \subset Z_5$, $Z_6 \subset Z_3$, we have $Y_5^\alpha \cap Z_5 \neq \varnothing$ and $Y_3^\alpha \cap Z_3 \neq \varnothing$. Therefore the following conditions are hold:

$$Y_7^\alpha \supseteq Z_7,\ Y_6^\alpha \supseteq Z_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5,\ Y_6^\alpha \cup Y_3^\alpha \supseteq Z_3,$$
$$Y_5^\alpha \cap Z_5 \neq \varnothing,\ Y_3^\alpha \cap Z_3 \neq \varnothing.$$

The Theorem is proved.

**Lemma 3. 1.** *The number of automorphisms of those semilattices, which are defined by the diagrams 1), 2), 3), 4), 8), 13), 14) and 15) in fig. 4 is equal to 1, those semilattices which are defined by the diagrams 5), 6), 7), 9), 10) 11) and 16) in fig. 4 is equal to 2 and that semilattice which is defined by the diagram 12) in fig. 4 is equal to 4.*

Proof: Let us prove the given lemma in case of the semilattice which is defined by the diagram 16) in fig. 4. The other cases we can prove analogously.

Suppose $Q = \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ (see Fig. 5) which satisfies the following conditions:

$$T_7 \subset T_5 \subset T_2 \subset T_0,\ T_7 \subset T_4 \subset T_2 \subset T_0,\ T_7 \subset T_4 \subset T_1 \subset T_0,$$
$$T_6 \subset T_3 \subset T_1 \subset T_0,\ T_6 \subset T_4 \subset T_2 \subset T_0,\ T_6 \subset T_4 \subset T_1 \subset T_0. \quad (3.1)$$

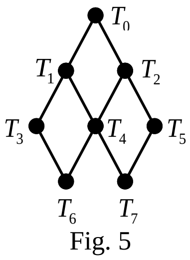

Fig. 5

Diagram of the Semilattice $Q$.

We prove that the number of all automorphisms of the given semilattice $Q$ is equal to 2. Indeed, if $T_i(n_i, m_i)$ denote the element $T_i$ of the semilattice $Q$ such that $n_i = \left|Q_{T_i}\right|$, $m_i = \left|\ddot{Q}_{T_i}\right|$ (see (1.1)) and $\varphi$ is arbitrary automorphism of the semilattice $Q$, then $\varphi(T_i) = T_j$ only if $n_i = n_j$ and $m_i = m_j$, i.e. $(n_i, m_i) = (n_j, m_j)$ $(3.2)$.

For the semilattice $Q$ we have:

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$$T_0 = T_0(1,8),\ T_1 = T_1(2,5),\ T_2 = T_2(2,5),\ T_3 = T_3(3,2),$$
$$T_4 = T_4(4,3),\ T_5 = T_5(3,2),\ T_6 = T_6(6,1),\ T_7 = T_7(6,1). \quad (3.3)$$

Therefore, $\varphi(T_0) = T_0$ and $\varphi(T_4) = T_4$ .

By definition of the semilattice $Q$ the $T_7$ and $T_6$ are minimal elements of the semilattice $Q$. According to (3.3), we can consider only two cases: $\varphi(T_7) = T_7$, $\varphi(T_6) = T_6$ or $\varphi(T_7) = T_6$, $\varphi(T_6) = T_7$. Let us consider each of them.

**a**) Suppose $\varphi(T_7) = T_7$ and $\varphi(T_6) = T_6$ . By definition of the semilattice $Q$ the elements $T_5$, $T_4$ are the covering elements of $T_7$ and the elements $T_4$, $T_3$ are the covering elements of $T_6$. Every automorphism corresponds covering element to covering element. But as we have $\varphi(T_4) = T_4$ , therefore $\varphi(T_5) = T_5$ and $\varphi(T_3) = T_3$. In addition, $T_2$ and $T_1$ are the covering elements of $T_5$ and $T_3$, respectively. So, we have $\varphi(T_2) = T_2$ and $\varphi(T_1) = T_1$. So, in our case we have

$$\varphi = \begin{pmatrix} T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \\ T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \end{pmatrix},$$

i.e. $\varphi$ is identical automorphism.

**b**) Suppose $\varphi(T_7) = T_6$ and $\varphi(T_6) = T_7$. If we use equalities $(3.3)$ and follow the proving steps of (**a**) case it gives the folowing: $\varphi(T_5) = T_3$, $\varphi(T_3) = T_5$ and $\varphi(T_2) = T_1$, $\varphi(T_1) = T_2$.

So, $\varphi = \begin{pmatrix} T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \\ T_0 & T_2 & T_1 & T_5 & T_4 & T_3 & T_7 & T_6 \end{pmatrix}$ is the second automorphism of the semilattice $Q$ .

Therefore the number of automorphisms of the given semilattice $Q$ is equal to 2.

The Lemma is proved.

**Theorem3.1.** *For any idempotent binary relation* $\varepsilon$ *of the semigroup* $B_X(D)$, *the order a subgroup* $G_X(D,\varepsilon)$ *of the semigroup* $B_X(D)$ *is one, or two, or four.*

*Proof.* We consider the following cases:

**a**) $Z_7 \cap Z_6 \neq \varnothing$;

**b**) $Z_7 \cap Z_6 = \varnothing$, $Z_7 \cap Z_3 \neq \varnothing$, $Z_6 \cap Z_5 \neq \varnothing$;

**c**) $Z_7 \cap Z_6 = \varnothing$, $Z_7 \cap Z_3 = \varnothing$, $Z_6 \cap Z_5 \neq \varnothing$;

**d**) $Z_7 \cap Z_6 = \varnothing$, $Z_7 \cap Z_3 \neq \varnothing$, $Z_6 \cap Z_5 = \varnothing$;

**e**) $Z_7 \cap Z_6 = \varnothing$, $Z_7 \cap Z_3 = \varnothing$, $Z_6 \cap Z_5 = \varnothing$, $Z_5 \cap Z_3 \neq \varnothing$;

**f**) $Z_5 \cap Z_3 = \varnothing$.

The Theorem we proved for the case **f**) $Z_5 \cap Z_3 = \varnothing$. The cases $a) - e)$ we can proved analogously.

Let $\varepsilon$ be an arbitrary binary relation of the semigroup $B_X(D)$. Now, if we denote by $\Phi$ the group of all complete automorphisms of the semilattice $V(D,\varepsilon)$, then by virtue of Theorem 1.3 we have that the groups $G_X(D,\varepsilon)$ and $\Phi$ are anti-isomorphic.

To prove the theorem, we will consider the following cases with regard to the idempotent binary relation $\varepsilon$ :

**1)** The idempotent binary relation $\varepsilon$ satisfies the conditions $1)-4)$, $8)$, $13)$, $14)$ and $15)$ of the Theorem 2.6, then the diagram of the semilattice $V(D,\varepsilon)$ has form 1, 2, 3, 4, 8, 13, 14 and 15 in Fig. 4. Therefore in this

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case the number of automorphisms of the semilattice $V(D,\varepsilon)$ is equal to one (see Lemma 3.1). Now, taking into account Theorem 1.3, we obtain $|G_X(D,\varepsilon)|=1$.

**2)** The idempotent binary relation $\varepsilon$ satisfies the conditions $5)-7)$, $9)-11)$ and $16)$ of the Theorem 2.6, then the diagram of the semilattice $V(D,\varepsilon)$ has form 5, 6, 7, 9, 10, 11 and 16 in Fig. 4. Therefore in this case the number of automorphisms of the semilattice $V(D,\varepsilon)$ is equal to two (see Lemma 3.1). Now, taking into account Theorem 1.3, we obtain $|G_X(D,\varepsilon)|=2$.

**3)** The idempotent binary relation $\varepsilon$ satisfies the condition $12)$ of the Theorem 2.6, then the diagram of the semilattice $V(D,\varepsilon)$ has form 12 in Fig. 4. Clearly, in this case the number of automorphisns of the semilattice $V(D,\varepsilon)$ is four (see Lemma 3.1). Now in view of Theorem 1.3, we obtain $|G_X(D,\varepsilon)|=4$.

Since the diagrams shown in Fig. 4 exhaust all the diagrams of the $XI-$subsemilattices of the semilattice $D$, when $Z_5 \cap Z_3 = \varnothing$. The idempotent binary relations of the semigroup $B_X(D)$ are exhausted by conditions $1)-16)$ of the Theorem 2.6. Hence it follows that for any idempotent $\varepsilon$ of the semigroup $B_X(D)$, the order a subgroup $G_X(D,\varepsilon)$ of the semigroup $B_X(D)$ is one, or two, or four.

The Theorem is proved.

## REFERENCES

[1] Ya. Diasamidze, Sh. Makharadze. Complete Semi groups of binary relations. Monograph. Kriter, Turkey, 2013, 1-520 pp.

[2] Ya. Diasamidze, Sh. Makharadze. Complete Semigroups of binary relations. Monograph. M., Sputnik+, 2010, 657 p. (Russian).

[3] Ya. I. Diasamidze. Complete Semigroups of Binary Relations. Journal of Mathematical Sciences, Plenum Publ. Cor., New York, Vol. 117, No. 4, 2003, 4271-4319.

[4] Diasamidze Ya., Makharadze Sh., Rokva N., on $XI-$semilattices of union. Bull. Georg. Nation. Acad. Sci., 2, № 1. 2008, 16-24.

[5] Diasamidze Ya., Makharadze Sh., Diasamidze Il., Idempotent and regular elements of complete semigroups of binary relations. Journal of Mathematical Sciences, Plenum Publ. Cor., New York, 153, № 4, 2008, 481-499.

[6] Diasamidze Ya., Makharadze Sh., Maximal subgroups of complete semigroups of binary relations. Proc. A. Razmadze Math. Inst. 131, 2003, 21-38.