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A Laplace type problem with non uniform distribution

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Abstract - In some previous papers the authors consider some Laplace type problem for different lattice, in particular in [7] the authors consider a Laplace type problem for a trapezoidal lattice with rectangle body test. In this paper we consider a lattice with fundamental cell composed by a trapetium but for the first time we consider as body test a random rectangle not uniformly distributed. We compute the probability that a random rectangles of constant sides intersects the a side of lattice when the position of rectangle is a random variable with exponential and $\gamma(2)$ distibution.

Keywords: **Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry.**

I. INTRODUCTION

In 1773, at a meeting of the Acadèmic des Sciences de Paris, Buffon posed a problem that later on should become knows as the famous " Buffon needle problem" : in a room, the floor of which is merely divided by parallel lines, at a distance a apart, a needle of length $l < a$ is allowed to fall at random: which is the probability that the needle intersects one of the lines? The solution, determined by Buffon by means of empirical methods, was $p = \frac{2l}{\pi a}$. The problem and its solution were published in 1777, in the Comptes rends de l'Acadèmie des sciences de Paris" .

In 1812, Laplace extended the problem by considering a room paved with equal tiles, shaped as rectangles of sides a and b, with $l < min(a,b)$ The solution was $p = \frac{2l(a+b)-l^2}{\pi ab}$, and it is obvious that the probability of Buffon can be obtained from that of Laplace by letting $b \to +\infty$.

This his first work on the needle problem was ignored for a long time until it was rediscovered in 1869 by the english mathematician W. Morgan Crofton, which will have a central role in the structuring of geometry or geometric probability, in which, to simplify, the calculation of the probability of a certain event has a geometric interpretation, in the sense that the set of possible events will have as image a certain figure F of the representational space, while the set of events will be favorable to picture a figure F_1 contained in F, and these figures will have to make the measurement.

K. Baclawski, M. Cerasoli and G.C Rota, in their book "Introduction to Probability" [1], have found that Buffon's needle problem has done so much to discuss the mathematicians of the nineteenth century due to the difficulty of making rigorous, and because it involves the presence of probability. In fact, it gives a technique with which you might find probabilistically (as was done) an approximate value of p.

In Particular, from 1974, several authors have shown different and innovative characterizations to this type of problem, they have considered several extensions in different directions: other types of regular and irregular lattices; other type test bodies; other spaces with dimension higher that two.

We restate now these problems in a slightly different form, wide will be useful for several different extensions: Let R be a lattice in the plane, with elementary cell C, and let T be a compact convex set, with fixed sharpe and dimensions, but random position in the plane (test body). As a natural assumption of randomness, we consider T as uniformly distributed in a bounded region of the plane. We also suppose that there exsists at least one postition of T such that T is entirely contained in C.

The problem is to find which is the probability that the body T intersects the boundary of the cells C of the lattice R.

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In Buffon's needle ploblem the cell C is a strip of breadth a, in the problem of Laplace $C$is a rectangle of sides a and b and in both cases T is a segment of length l.The study of Buffon-Laplace problems for lattices with obstacles is of interest in many folds, as such lattices are useful in modelling semples of biological tissue, in the geometric distribution of vegetation, in transport and problem of traffic, in geological structures, in crystallography, in problems of quality in porous materials, in topography and in many other fields.

In [2], [3], [4], [5], [6], [7] and [8] the authors consider several different Buffon-Laplace type problem. Now, for the first time, we consider a Laplace type problem for a fundamental cell composed by a trapetium considering as body test a random rectangle non uniform distibuted.

II. MAIN RESULTS

Let $\Re(a,b;\alpha)$ a lattice with the fundamental cell C_0 an trapetium of sides $a<b$, with $\frac{\pi}{4}\leq\alpha\leq\frac{\pi}{3}$

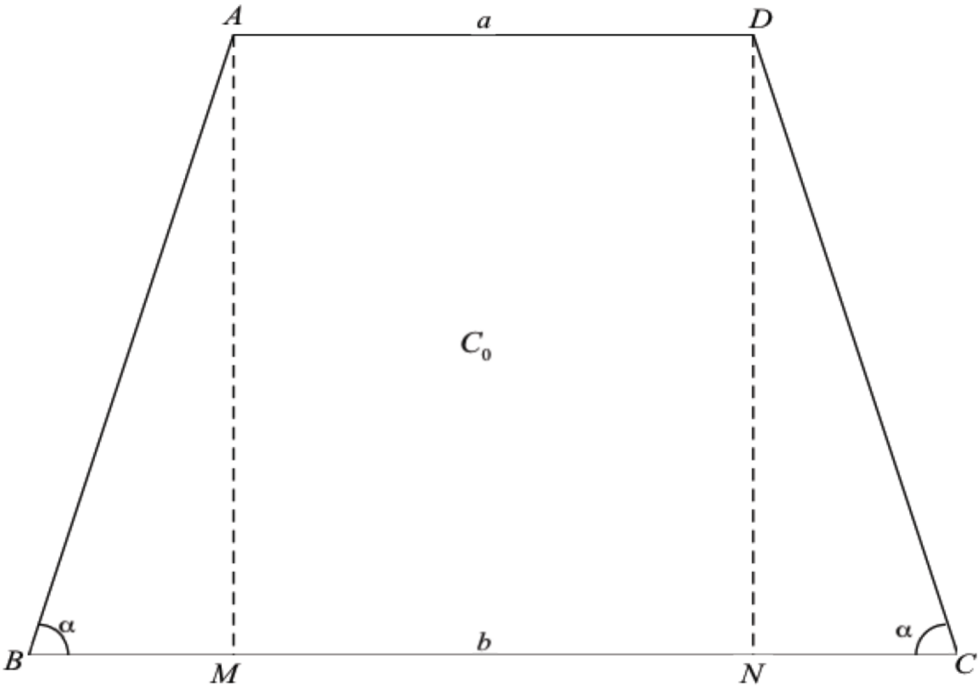

Fig.1

By fig. 1 we have that:

$$areaC_0=\frac{(b^2-a^2)tg\alpha}{4}. \tag{1}$$

Theorem 1*The probability that a random rectangle* r *of constant sides* l,m *with* $0<m\leq l<\frac{a}{2}$ *and intersects a side of the lattice* $\Re$ *is:*

$$P_{int}=\frac{2}{(b^2-a^2)tg\alpha\int_0^{\alpha}f(\varphi)d\varphi}\cdot$$

$$\int_0^{\alpha}\{(a+b)(l\sin\varphi+m\cos\varphi)+2(b-a)tg\alpha(l\cos\varphi+m\sin\varphi)-$$

$$\frac{l^2}{\sin\alpha}[\sin\alpha\sin2\varphi+(1+\cos\alpha)(1-\cos2\varphi)]-$$

$$m^2\sin2\varphi+\frac{lm}{2\sin\alpha}[\cos2\varphi-(1+\cos\alpha)\sin2\varphi]+\frac{3}{2}lm\Big\}f(\varphi)d\varphi, \tag{2}$$

where φ is the angle formed by the side of lenght l of the rectangle r with the line BC (or AD), the

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position of r is determined by the your center and by the angle φ.

Proof. We consider the limiting positions of the r, for a specified value of φ, in the cell C_0. We obtain the figure

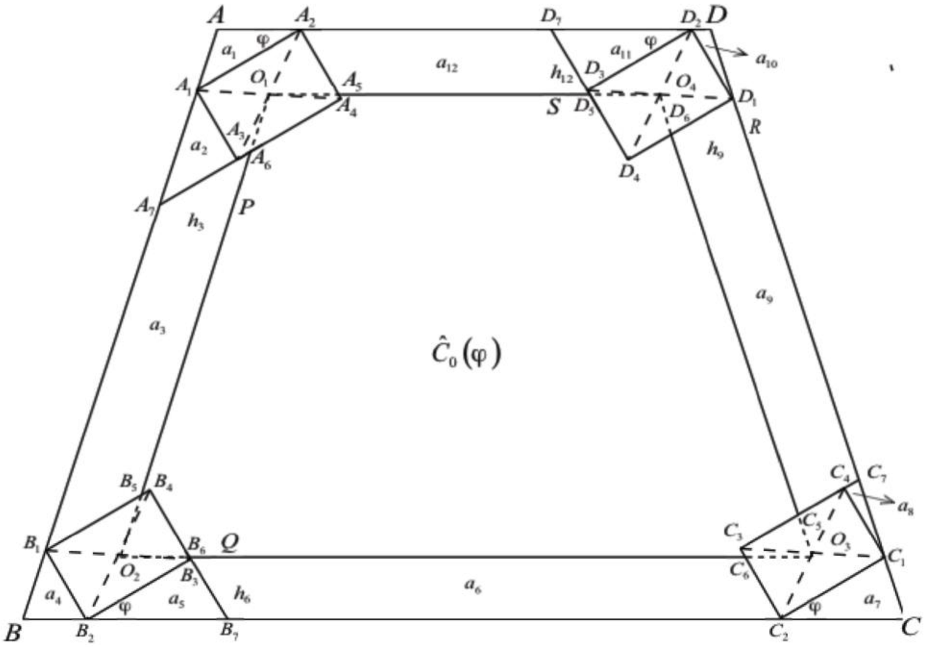

fig.2

and the formula

$$area\hat{C}_0(\varphi) = areaC_0 - 4lm - \sum_{i=1}^{12} areaa_i(\varphi). \quad (3)$$

By fig. 2 we have that:

$$areaa_1(\varphi) = \frac{l^2 \sin\varphi \sin(\alpha - \varphi)}{2\sin\alpha},$$

$$areaa_2(\varphi) = \frac{m^2}{2} ctg(\alpha - \varphi),$$

$$areaa_3(\varphi) = \frac{1}{2}[l\sin(\alpha - \varphi) + m\cos(\alpha - \varphi)]\left[\frac{b-a}{\cos\alpha} - \frac{l\sin\varphi + m\cos\varphi}{\sin\alpha}\right] -$$

$$\frac{1}{2}[lm + m^2 ctg(\alpha - \varphi)],$$

$$areaa_4(\varphi) = \frac{m^2 \cos\varphi \cos(\alpha - \varphi)}{2\sin\alpha},$$

$$areaa_5(\varphi) = \frac{l^2}{2} tg\alpha,$$

$$areaa_6(\varphi) = \frac{1}{2}(l\sin\varphi + m\cos\varphi)\left[b - \frac{m\cos(\alpha - \varphi) + l\sin(\varphi + \alpha)}{\sin\alpha}\right] - \frac{1}{2}(l^2 tg\varphi + lm),$$

$$areaa_7(\varphi) = \frac{l^2 \sin\varphi \sin(\varphi + \alpha)}{2\sin\alpha},$$

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$$areaa_8(\varphi) = -\frac{m^2}{2} ctg(\alpha + \varphi),$$

$$areaa_9(\varphi) = \frac{1}{2}[l\sin(\varphi + \alpha) - m\cos(\varphi + \alpha)]\left[\frac{b-a}{2\cos\alpha} - \frac{l\sin\varphi + m\cos\varphi}{\sin\alpha}\right] +$$

$$\frac{m^2}{2} ctg(\varphi + \alpha) - \frac{lm}{2},$$

$$areaa_{10}(\varphi) = -\frac{m^2 \cos\varphi\cos(\varphi+\alpha)}{2\sin\alpha},$$

$$areaa_{11}(\varphi) = \frac{l^2}{2} tg\varphi,$$

$$areaa_{12}(\varphi) = \frac{1}{2}(l\sin\varphi + m\cos\varphi)\left[a - \frac{l\sin\varphi - m\cos(\varphi+\alpha)}{\sin\alpha}\right] - \frac{l^2}{2} tg\varphi - \frac{lm}{2}.$$

Replacing these relations in (3) follow that:

$$area\hat{C}_0(\varphi) = areaC_0 -$$

$$\left\{\frac{1}{2}(a+b)(l\sin\varphi + m\cos\varphi) + (b-a)tg\alpha(l\cos\varphi + m\sin\varphi) -\right.$$

$$\frac{l^2}{2\sin\alpha}[\sin\alpha\sin2\varphi + (1+\cos\alpha)(1-\cos2\varphi)] - \frac{m^2}{2}\sin2\varphi +$$

$$\left.\frac{lm}{4\sin\alpha}[\cos2\varphi - (1+\cos\alpha)\sin2\varphi] + \frac{3}{4}lm\right\}. \quad (4)$$

Denoting by M, the set of the all rectangles r which have their center in the cell C_0 denote likewise by N the set of the all rectangles r completely contained in C_0. In view of [10], we get:

$$P_{int} = 1 - \frac{\mu(N)}{\mu(M)}, \quad (5)$$

where μ is the Lebesgue measure in Euclidean plane.
To compute the above measures we use the Poincaré kinematic measure [9]:

$$dK = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinate of the center of r and φ the angle already defined.

Considering that the direction of r is a random variable with density of probability $f(\varphi)$, we have that:

$$\mu(M) = \int_0^\alpha f(\varphi)d\varphi \iint_{\{(x,y)\in C_0\}} dxdy = \int_0^\alpha (areaC_0)f(\varphi)d\varphi = areaC_0\int_0^\alpha f(\varphi)d\varphi, \quad (6)$$

and

$$\mu(N) = \int_0^\alpha f(\varphi)d\varphi \iint_{\{(x,y)\in\hat{C}_0(\varphi)\}} dxdy = \int_0^\alpha \left(area\hat{C}_0(\varphi)\right)f(\varphi)d\varphi =$$

$$areaC_0\int_0^\alpha f(\varphi)d\varphi -$$

$$\int_0^\alpha \left\{\frac{1}{2}(a+b)(l\sin\varphi + m\cos\varphi) + (b-a)tg\alpha(l\cos\varphi + m\sin\varphi) -\right.$$

$$\frac{l^2}{2\sin\alpha}[\sin\alpha\sin2\varphi + (1+\cos\alpha)(1-\cos2\varphi)] - \frac{m^2}{2}\sin2\varphi +$$

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$$\frac{lm}{4\sin\alpha}[\cos 2\varphi-(1+\cos\alpha)\sin 2\varphi]+\frac{3}{4}lm\Big\} f(\varphi)d\varphi, \quad (7)$$

then

$$P_{int}=\frac{2}{(b^2-a^2)tg\alpha\int_0^\alpha f(\varphi)d\varphi}\cdot$$

$$\int_0^\alpha \{(a+b)(l\sin\varphi+m\cos\varphi)+2(b-a)tg\alpha(l\cos\varphi+m\sin\varphi)-$$

$$\frac{l^2}{\sin\alpha}[\sin\alpha\sin 2\varphi+(1+\cos\alpha)(1-\cos 2\varphi)]-m^2\sin 2\varphi+$$

$$\frac{lm}{2\sin\alpha}[\cos 2\varphi-(1+\cos\alpha)\sin 2\varphi]+\frac{3}{2}lm\Big\} f(\varphi)d\varphi. \quad (8)$$

III. ESPONENTIAL RANDOM VARIABLE

Considering

$$f(\varphi)=e^{-\varphi},$$

and by the change of variable $e^{-\varphi}=u$, we obtain that:

$$\int_0^\alpha f(\varphi)d\varphi=1-e^{-\alpha}. \quad (9)$$

In the same way, we have that:

$$\int_0^\alpha e^{-\varphi}\sin\varphi d\varphi=\frac{1}{2}-\frac{1}{2}e^{-\alpha}(\sin\alpha+\cos\alpha),$$

$$\int_0^\alpha e^{-\varphi}\cos\varphi d\varphi=\frac{1}{2}+\frac{1}{2}e^{-\alpha}(\sin\alpha-\cos\alpha), \quad (10)$$

and

$$\int_0^\alpha e^{-\varphi}\sin 2\varphi d\varphi=\frac{1}{5}[2-e^{-\alpha}(\sin 2\alpha+2\cos 2\alpha)]$$

$$\int_0^\alpha e^{-\varphi}\cos 2\varphi d\varphi=\frac{1}{5}[1+e^{-\alpha}(2\sin 2\alpha-\cos 2\alpha)]. \quad (11)$$

Replacing in (9) the relations (10) and (11) we obtain the following:

Theorem 2*The probability that a random rectangle* r *of constant sides* l, m *with* $0<m\le l<\frac{a}{2}$ *and distributed according the exponential distribution, intersects a side of the lattice* $\Re$ *is:*

$$P_{int}=\frac{2}{(b^2-a^2)tg\alpha(1-e^{-\alpha})}\cdot$$

$$\Big\{\frac{1}{2}[l(a+b)+2m(b-a)tg\alpha][1-e^{-\alpha}(\sin\alpha+\cos\alpha)]+$$

$$\frac{1}{2}[m(a+b)+2l(b-a)tg\alpha][1+e^{-\alpha}(\sin\alpha-\cos\alpha)]+\frac{1}{5}\left(l^2+m^2-\frac{1+\cos\alpha}{2\sin\alpha}lm\right)\cdot$$

$$[2-e^{-\alpha}(\sin 2\alpha+2\cos 2\alpha)]+\frac{1}{5}\left(\frac{1+\cos\alpha}{\sin\alpha}l^2+\frac{1}{2\sin\alpha}lm\right)$$

$$[1+e^{-\alpha}(2\sin 2\alpha-\cos 2\alpha)]+\frac{3}{2}lm(1-e^{-\alpha})\Big\}.$$

$\gamma(2)$ **random variable**

Considering now

$$f(\varphi)=\varphi e^{-\varphi},$$

we obtain that:

$$\int_1^{\alpha} \varphi e^{-\varphi} = 1 - (1+\alpha)e^{-\alpha},$$

$$\int_0^{\alpha} \varphi \sin\varphi e^{-\varphi} d\varphi = \frac{1}{2} - \frac{1}{2} e^{-\alpha}\cos\alpha - \frac{\alpha}{2} e^{-\alpha}(\sin\alpha + \cos\alpha),$$

$$\int_0^{\alpha} \varphi e^{-\varphi} \cos\varphi d\varphi = \frac{1}{2} e^{-\alpha}(\sin\alpha + \cos\alpha) + \frac{\alpha}{2} e^{-\alpha}(\sin\alpha - \cos\alpha)$$

and

$$\int_0^{\alpha} \varphi e^{-\varphi} \sin 2\varphi d\varphi = \frac{4}{5} + \frac{1}{25} e^{-\alpha}(3\sin 2\alpha + 8\cos 2\alpha) + \alpha e^{-\alpha}(\sin 2\alpha - 2\cos 2\alpha),$$

$$\int_0^{\alpha} \varphi e^{-\varphi} \cos 2\varphi d\varphi = \frac{1}{5} + \frac{1}{25} e^{-\alpha}(4\sin 2\alpha + 9\cos 2\alpha) + \alpha e^{-\alpha}(\sin 2\alpha - \cos 2\alpha).$$

We have that:

Theorem 3 *The probability that a random rectangle* r *of constant sides* l, m *with* $0 < m \leq l < \frac{a}{2}$ *and distributed according the* $\gamma(2)$ *distribution, intersects a side of the lattice* $\Re$ *is:*

$$P_{int} = \frac{2}{(b^2 - a^2) tg\alpha (1 - e^{-\alpha})} \cdot$$

$$\Big\{ \frac{1}{2} [l(a+b) + 2m(b-a)tg\alpha][1 - e^{-\alpha}\cos\alpha - \alpha e^{-\alpha}(\sin\alpha + \cos\alpha)] +$$

$$\frac{1}{2} e^{-\alpha} [m(a+b) + 2l(b-a)tg\alpha][\sin\alpha + \cos\alpha + \alpha(\sin\alpha - \cos\alpha)] -$$

$$\left(l^2 + m^2 - \frac{1+\cos\alpha}{2\sin\alpha} lm\right) \cdot$$

$$\left[\frac{4}{5} + \frac{1}{25} e^{-\alpha}(3\sin 2\alpha + 8\cos 2\alpha) + \alpha e^{-\alpha}(\sin 2\alpha - 2\cos 2\alpha)\right] +$$

$$\left(\frac{1+\cos\alpha}{\sin\alpha} l^2 + \frac{1}{2\sin\alpha} lm\right) \cdot$$

$$\left[\frac{1}{5} + \frac{1}{25} e^{-\alpha}(4\sin 2\alpha + 9\cos 2\alpha) + \alpha e^{-\alpha}(\sin 2\alpha - \cos 2\alpha)\right] -$$

$$\left(\frac{1+\cos\alpha}{\sin\alpha} l^2 - \frac{3}{2} lm\right) [1 - (1+\alpha)e^{-\alpha}]\Big\}.$$

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