ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 4, July 2015

On Some Properties of Regular Elements of Complete Semi groups Defined by Semi lattices of the Class $\Sigma_4(X,8)$

Yasha Diasamidze1 and Alexander Bakuridze[2]
Faculty of Physics, Mathematics and Computer Sciences, Department of Mathematics,
Shota Rustaveli Batumi State University,
35 Ninoshvili St., Batumi 6010, Georgia

ABSTRACT: In this article, we prove that all regular elements R_D of the complete semi group $B_X(D)$ defined by an $X-$semi lattice D of the class $\Sigma_4(X,8)$ are subsemigroups of this semigroup.

AMS Subject Classification: 20M05
Key words and phrases: Semi lattice, semi group, binary relation.

I. INTRODUCTION

Let X be an arbitrary nonempty set, D be an $X-$semi lattice of unions, i.e. such a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping of the set X in the set D. To each mapping f we put into correspondence a binary relation α_f on the set X that satisfies the condition

$$\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x)).$$

The set of all such α_f ($f: X\to D$) is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semi group with respect to the operation of multiplication of binary relations, which is called a complete semi group of binary relations defined by an $X-$semi lattice of unions D.

We denote by $\varnothing$ an empty binary relation or an empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Further, let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $\varnothing\neq D'\subseteq D$, $\breve{D}=\bigcup_{Y\in D} Y$ and $t\in\breve{D}$. We denote by the symbols $y\alpha$, $Y\alpha$, $V(D,\alpha)$, X^*, $V(X^*,\alpha)$ and D'_t the following sets:

$$y\alpha=\{x\in X \mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y} y\alpha,\ V(D,\alpha)=\{Y\alpha \mid Y\in D\},$$
$$X^*=\{Y \mid \varnothing\neq Y\subseteq X\},\ V(X^*,\alpha)=\{Y\alpha \mid \varnothing\neq Y\subseteq X\},$$
$$D'_t=\{Z'\in D' \mid t\in Z'\}.$$

We use the symbol $\Lambda(D,D')$ to denote the exact lower bound of the set D' in the semilattice D.

Definition 1. We say that the complete $X-$semilattice of unions D is an $XI-$semilattice of unions if it satisfies the following two conditions:

a) $\Lambda(D,D_t)\in D$ for any $t\in\breve{D}$;

b) $Z=\bigcup_{t\in Z}\Lambda(D,D_t)$ for any nonempty element Z of the semilattice D. (see [1] and [2]).

Let X and $\Sigma_4(X,8)$ be respectively an arbitrary nonempty set and the class of $X-$semilattices of unions, where each element is isomorphic to some $X-$semilattice of unions $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ that satisfies the conditions

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 4, July 2015

$$\begin{aligned}
&Z_7 \subset Z_4 \subset Z_1 \subset \breve{D},\ Z_7 \subset Z_6 \subset Z_1 \subset \breve{D},\\
&Z_7 \subset Z_6 \subset Z_3 \subset \breve{D},\ Z_1 \setminus Z_2 \neq \varnothing,\ Z_2 \setminus Z_1 \neq \varnothing,\\
&Z_1 \setminus Z_3 \neq \varnothing,\ Z_3 \setminus Z_1 \neq \varnothing,\ Z_2 \setminus Z_3 \neq \varnothing,\ Z_3 \setminus Z_2 \neq \varnothing,\\
&Z_4 \setminus Z_5 \neq \varnothing,\ Z_5 \setminus Z_4 \neq \varnothing,\ Z_4 \setminus Z_6 \neq \varnothing,\ Z_6 \setminus Z_4 \neq \varnothing,\\
&Z_5 \setminus Z_6 \neq \varnothing,\ Z_6 \setminus Z_5 \neq \varnothing,\ Z_1 \cup Z_2 = Z_1 \cup Z_3 = \breve{D},\\
&Z_2 \cup Z_3 = Z_4 \cup Z_2 = Z_4 \cup Z_3 = Z_5 \cup Z_2 = Z_5 \cup Z_3 = \breve{D},\\
&Z_4 \cup Z_5 = Z_4 \cup Z_6 = Z_5 \cup Z_6 = Z_1.
\end{aligned}$$

An $X-$semilattice that satisfies conditions (1) is shown in Fig.1.

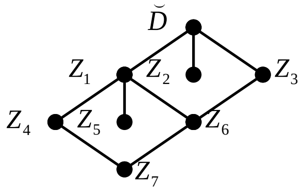

Fig.1

For the complete semigroups of binary relations defined by semilattices of the class $\Sigma_4(X,8)$ the following statements (see [7, 8]) are well known.

Lemma 1. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_4(X,8)$. Then the following sets exhaust all $XI-$subsemilattices of the semilattice D:

1) $\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}$ (see diagram 1 in Fig. 2);

2) $\{Z_7,Z_6\},\{Z_7,Z_4\},\{Z_7,Z_3\},\{Z_7,Z_1\},\{Z_7,\breve{D}\},\{Z_6,Z_3\},\{Z_6,Z_1\},$
$\{Z_6,\breve{D}\},\{Z_5,Z_1\},\{Z_5,\breve{D}\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},$
$\{Z_1,\breve{D}\}$
(see diagram 2 in Fig. 2);

3) $\{Z_7,Z_6,\breve{D}\},\{Z_7,Z_4,\breve{D}\},\{Z_7,Z_3,\breve{D}\},\{Z_7,Z_1,\breve{D}\},\{Z_5,Z_1,\breve{D}\},\{Z_7,Z_6,Z_3\},$
$\{Z_7,Z_6,Z_1\},\{Z_7,Z_4,Z_1\},\{Z_6,Z_3,\breve{D}\},\{Z_6,Z_1,\breve{D}\},\{Z_4,Z_1,\breve{D}\}$
(see diagram 3 in Fig. 2);

4) $\{Z_7,Z_6,Z_3,\breve{D}\},\{Z_7,Z_6,Z_1,\breve{D}\},\{Z_7,Z_4,Z_1,\breve{D}\}$ (see diagram 4 in Fig. 2);

5) $\{Z_7,Z_6,Z_4,Z_1\},\{Z_7,Z_4,Z_3,\breve{D}\},\{Z_6,Z_3,Z_1,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\}$
(see diagram 5 in Fig. 2);

6) $\{Z_7,Z_6,Z_4,Z_1,\breve{D}\}$ (see diagram 6 in Fig. 2);

7) $\{Z_7,Z_6,Z_3,Z_1,\breve{D}\}$ (see diagram 7 in Fig. 2);

8) $\{Z_7,Z_6,Z_4,Z_3,Z_1,\breve{D}\}$ (see diagram 8 in Fig. 2).

Fig. 2 gives the diagrams of all $XI-$subsemilattices of D.

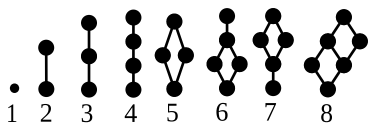

Fig. 2

Definition 2. Let $\beta\in B_X(D)$. If $\beta\circ\delta\circ\beta=\beta$ for some $\delta\in B_X(D)$, then a binary relation β is called a regular element of the semi group $B_X(D)$.

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 4, July 2015

Definition 3. The one-to-one mapping φ between the complete $X-$semilattices of unions D' and D'' is called a complete isomorphism if the condition

$$\varphi(\cup D_1)=\bigcup_{T'\in D_1}\varphi(T')$$

is fulfilled for each nonempty subset D_1 of the semilattice D' (see [1], Definition 6.3.2).

Definition 4. Let α be some binary relation of the semi group $B_X(D)$. We say that a complete isomorphism φ between the complete semilattices of unions Q and D' is a complete $\alpha-$isomorphism if

a) $Q=V(D,\alpha)$;

b) $\varphi(\varnothing)=\varnothing$ for $\varnothing\in V(D,\alpha)$ and $\varphi(T)\alpha=T$ for any $T\in V(D,\alpha)$ (see [1, Definition 6.3.3]).

Definition 5. Let D be an arbitrary complete $X-$ semilattice of unions, $\alpha\in B_X(D)$ and $Y_T^\alpha=\{x\in X\mid x\alpha=T\}$. If

$$V[\alpha]=\begin{cases}V(X^*,\alpha),\ if\ \varnothing\notin D,\\ V(X^*,\alpha),\ if\ \varnothing\in V(X^*,\alpha),\\ V(X^*,\alpha)\cup\{\varnothing\},\ if\ \varnothing\notin V(X^*,\alpha)\ and\ \varnothing\in D,\end{cases}$$

then it obviously follows that any binary relation α of the semi group $B_X(D)$ can always be represented in the form $\alpha=\bigcup_{T\in V[\alpha]}(Y_T^\alpha\times T)$.

In the sequel we will call such a representation of a binary relation α quasinormal.

Note that for a quasinormal representation of a binary relation α, it is not all sets Y_T^α that may differ from an empty set. But for such a representation the following conditions are always fulfilled:

a) $Y_T^\alpha\cap Y_{T'}^\alpha=\varnothing$ for any $T,T'\in D$ and $T\neq T'$;

b) $X=\bigcup_{T\in V[\alpha]}Y_T^\alpha$

(see [1], 1.11).

Theorem 1. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_4(X,8)$. Then a binary relation α of the semigroup $B_X(D)$ with a quasinormal representation in one of the forms given below is a regular element of this semigroup iff there exists a complete $\alpha-$isomorphism φ of the semilattice $V(D,\alpha)$ on some subsemilattice D' of the semilattice D that satisfies at least one of the following conditions:

a) $\alpha=X\times T$ for some $T\in D$;

b) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')$ for some $T,T'\in D$, $T\subset T'$ and $Y_T^\alpha,Y_{T'}^\alpha\notin\{\varnothing\}$, which satisfies the conditions: $Y_T^\alpha\supseteq\varphi(T)$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$;

c) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')$ for some $T,T',T''\in D$, $T\subset T'\subset T''$, and $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$,which satisfies the conditions: $Y_T^\alpha\supseteq\varphi(T)$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_{T''}^\alpha\cap\varphi(T'')\neq\varnothing$;

d) $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_0^\alpha\times\breve{D})$ for some $T,T',Z\in D$, $Z_7\subset T\subset T'\subset\breve{D}$, $Y_7^\alpha,Y_T^\alpha,Y_{T'}^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and $Y_7^\alpha\supseteq\varphi(Z_7)$, $Y_7^\alpha\cup Y_T^\alpha\supseteq\varphi(T)$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_T^\alpha\cap\varphi(T)\neq\varnothing$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_0^\alpha\cap\varphi(\breve{D})\neq\varnothing$;

e) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_Z^\alpha\times Z)\cup(Y_{T'\cup Z}^\alpha\times(T'\cup Z))$ for some $T,T',Z\in D$, $T\subset T'$, $T\subset Z$, $T'\setminus Z\neq\varnothing$, $Z\setminus T'\neq\varnothing$, $Y_T^\alpha,Y_{T'}^\alpha,Y_Z^\alpha\notin\{\varnothing\}$ and $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq\varphi(T')$, $Y_T^\alpha\cup Y_Z^\alpha\supseteq\varphi(Z)$, $Y_{T'}^\alpha\cap\varphi(T')\neq\varnothing$, $Y_Z^\alpha\cap\varphi(Z)\neq\varnothing$;

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 4, July 2015

f) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$ for some $Y_7^\alpha,Y_6^\alpha,Y_4^\alpha,Y_0^\alpha\notin\{\varnothing\}$,which satisfies the conditions: $Y_7^\alpha\cup Y_6^\alpha\supseteq\varphi(Z_6)$, $Y_7^\alpha\cup Y_4^\alpha\supseteq\varphi(Z_4)$, $Y_6^\alpha\cap\varphi(Z_6)\neq\varnothing$, $Y_4^\alpha\cap\varphi(Z_4)\neq\varnothing$, $Y_0^\alpha\cap\varphi(\breve{D})\neq\varnothing$.

g) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$ for some $Y_7^\alpha,Y_6^\alpha,Y_3^\alpha,Y_1^\alpha\notin\{\varnothing\}$ which satisfies the conditions: $Y_7^\alpha\cup Y_6^\alpha\supseteq\varphi(Z_6)$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_3^\alpha\supseteq\varphi(Z_3)$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_1^\alpha\supseteq\varphi(Z_1)$, $Y_6^\alpha\cap\varphi(Z_6)\neq\varnothing$, $Y_3^\alpha\cap\varphi(Z_3)\neq\varnothing$, $Y_1^\alpha\cap\varphi(Z_1)\neq\varnothing$.

h) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$ for some $Y_7^\alpha,Y_6^\alpha,Y_4^\alpha,Y_3^\alpha\notin\{\varnothing\}$ which satisfies the conditions: $Y_7^\alpha\supseteq\varphi(Z_7)$, $Y_7^\alpha\cup Y_6^\alpha\supseteq\varphi(Z_6)$, $Y_7^\alpha\cup Y_4^\alpha\supseteq\varphi(Z_4)$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_3^\alpha\supseteq\varphi(Z_3)$, $Y_6^\alpha\cap\varphi(Z_6)\neq\varnothing$, $Y_4^\alpha\cap\varphi(Z_4)\neq\varnothing$, $Y_3^\alpha\cap\varphi(Z_3)\neq\varnothing$.

Theorem 2. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_4(X,8)$. Then the set R_D of all regular elements of the semigroup $B_X(D)$ is a subsemigroup of this semigroup.

Proof. Let $\alpha,\beta\in B_X(D)$. Then for a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form

$$\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right), \quad ...(2)$$

we have

$$V(D,\alpha\circ\beta)\subseteq V(D,\beta) \quad ...(3)$$

and

$$\alpha\circ\beta=\left(Y_7^\alpha\times Z_7\beta\right)\cup\left(Y_6^\alpha\times Z_6\beta\right)\cup\left(Y_5^\alpha\times Z_5\beta\right)\cup \\ \cup\left(Y_4^\alpha\times Z_4\beta\right)\cup\left(Y_3^\alpha\times Z_3\beta\right)\cup\left(Y_2^\alpha\times Z_2\beta\right)\cup\left(Y_1^\alpha\times Z_1\beta\right)\cup\left(Y_0^\alpha\times\breve{D}\beta\right). \quad ...(4)$$

Therefore

$$V(D,\alpha\circ\beta)=\{Z_7\beta,\ Z_6\beta,\ Z_5\beta,\ Z_4\beta,\ Z_3\beta,\ Z_2\beta,\ Z_1\beta,\ \breve{D}\beta\}. \quad ...(5)$$

Note that the mapping $\varphi_{\alpha\beta}:V(D,\alpha)\to V(D,\alpha\circ\beta)$, which satisfies the condition $\varphi_{\alpha\beta}(Z)=Z\beta$ for any element Z of the set $V(D,\alpha)$, is the monotone mapping of the semilattice $V(D,\alpha)$ on the semilattice $V(D,\alpha\circ\beta)$, i.e., $\varphi_{\alpha\beta}$ is the mapping for which the condition $Z\subseteq Z'$ always implies the inclusion $Z\beta\subseteq Z'\beta$ for each $Z,Z'\in V(D,\alpha)$. From inclusion (3) we have that the mapping $\varphi_{\alpha\beta}$ is the monotone mapping of the semilattice $V(D,\alpha)$ in the semilattice $V(D,\beta)$, i.e., if the semilattice $V(D,\alpha)$ is finite and its diagram is well known, then the diagram of the semilattice $V(D,\alpha\circ\beta)$ is a subdiagram of the diagram of the semilattice $V(D,\beta)$.

Now let the binary relations α and β be any regular elements of the semigroup $B_X(D)$. Then $V(D,\alpha)$ and $V(D,\beta)$ are $XI-$semilattices and by Theorem 1 the diagrams of all subsemilattices of D are given in Fig. 2. From these diagrams it follows that the semilattice $V(D,\alpha)$ has the smallest element. From equalities (3) and (5) it follows that the semilattice $V(D,\alpha\circ\beta)$ also has the smallest element. This fact immediately implies that the mapping $\varphi_{\alpha\beta}$ is the monotone mapping of the semilattice $V(D,\alpha)$ in the semilattice $V(D,\beta)$.

We consider the following case.

1) The diagram of the semilattice $V(D,\beta)$ has form 8 in Fig. 2 . From the definition of this semilattice D it follows that we have a unique subsemilattice $D'=\{Z_7,Z_6,Z_4,Z_3,Z_1,\breve{D}\}$ of the semilattice D, the diagram of which has form 8 in Fig. 2 (see Fig. 3). Any subsemilattice of this semilattice has a diagram of the form shown in Fig. 4.

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 4, July 2015

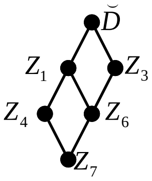

Fig.3

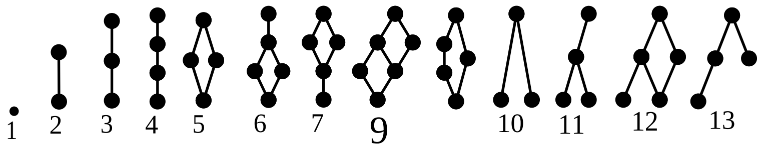

Fig. 4

By assumption, we have $V(D,\alpha\circ\beta)\subseteq V(D,\beta)$. So, the diagram of the semilattice $V(D,\alpha\circ\beta)$ can be one of the diagrams shown in Fig. 4. But by Theorem 1 the semilattice $V(D,\alpha\circ\beta)$ has the smallest element. From this it follows that the semilattice $V(D,\alpha\circ\beta)$ can have one of diagrams $1-9$ given in Fig. 4.

Let us prove that the diagram of the semilattice $V(D,\alpha\circ\beta)$ is always different from diagram 9 in Fig. 4. Indeed, diagram 9 is a subdiagram of diagram 3. Therefore, by the definition of the semilattice D, the following equalities are fulfilled:

$$V(D,\alpha)=\{Z_7,Z_6,Z_4,Z_3,Z_1,\breve{D}\} \text{ and } V(D,\alpha\circ\beta)=\{Z_7\beta,Z_6\beta,Z_4\beta,Z_3\beta,Z_1\beta,\breve{D}\beta\}\subseteq V(D,\beta).$$

Suppose that there exists some regular element β for which $Z\beta=Z'\beta$ for some $Z\beta,Z'\beta\in V(D,\alpha\circ\beta)$ and the other five elements are pairwise disjoint subsets of the set X.

We consider the following cases:

a) $Z,Z',Z''\in D$, $Z\subset Z'\subset Z''$, $Z\beta,Z'\beta,Z''\beta\in V(D,\alpha\circ\beta)$ and $Z\beta=Z''\beta$. Then $Z\beta=Z'\beta=Z''\beta$, i.e., in this case we have $|V(D,\alpha\circ\beta)|\le 4$. The inequality $|V(D,\alpha\circ\beta)|\le 4$ contradicts the equality $|V(D,\alpha\circ\beta)|=5$. So, we have $Z_7\beta\ne Z_3\beta,\ Z_7\beta\ne Z_1\beta,\ Z_6\beta\ne\breve{D}\beta,\ Z_4\beta\ne\breve{D}\beta.$

b) $Z_7\beta=Z_6\beta$. By the definition of the semilattice D we have $Z_6\cup Z_4=Z_1$ and $Z_4\supset Z_7$. From this it follows that $Z_6\beta\cup Z_4\beta=Z_1\beta$, $Z_4\beta\supseteq Z_7\beta=Z_6\beta$ and $Z_1\beta=Z_6\beta\cup Z_4\beta=Z_4\beta$. So, $|V(D,\alpha\circ\beta)|\le 4$. The inequality $|V(D,\alpha\circ\beta)|\le 4$ contradicts the equality $|V(D,\alpha\circ\beta)|=5$. Therefore $Z_7\beta\ne Z_6\beta$.

c) $Z_7\beta=Z_4\beta$. By the definition of the semilattice D we have $Z_6\cup Z_4=Z_1$ and $Z_6\supset Z_7$. Thus we have $Z_6\beta\cup Z_4\beta=Z_1\beta$, $Z_6\beta\supseteq Z_7\beta=Z_4\beta$ and $Z_1\beta=Z_6\beta\cup Z_4\beta=Z_6\beta$. So, $|V(D,\alpha\circ\beta)|\le 4$. The inequality $|V(D,\alpha\circ\beta)|\le 4$ contradicts the equality $|V(D,\alpha\circ\beta)|=5$. Therefore $Z_7\beta\ne Z_4\beta$.

d) $Z_6\beta=Z_1\beta$. By the definition of the semilattice D we have $Z_1\cup Z_3=\breve{D}$ and $Z_3\supset Z_6$, Thus we have $Z_1\beta\cup Z_3\beta=\breve{D}\beta$, $Z_3\beta\supseteq Z_6\beta$ and $Z_3\beta=\breve{D}\beta$. So, $|V(D,\alpha\circ\beta)|\le 4$. The inequality $|V(D,\alpha\circ\beta)|\le 4$ contradicts the equality $|V(D,\alpha\circ\beta)|=5$. Therefore $Z_6\beta\ne Z_1\beta$.

e) $Z_6\beta=Z_3\beta$. By the definition of the semilattice D we have $Z_1\cup Z_3=\breve{D}$ and $Z_1\supset Z_6$. Thus we have $Z_1\beta\cup Z_3\beta=\breve{D}\beta$, $Z_1\beta\supseteq Z_6\beta$ and $Z_1\beta=\breve{D}\beta$. So, $|V(D,\alpha\circ\beta)|\le 4$. The inequality $|V(D,\alpha\circ\beta)|\le 4$ contradicts the equality $|V(D,\alpha\circ\beta)|=5$. Therefore $Z_6\beta\ne Z_3\beta$.

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 4, July 2015

f) $Z_4\beta = Z_1\beta$. Since five elements of the semilattice $V(D,\alpha\circ\beta)$ are pairwise disjoint subsets of the set X, the following inclusions are true:

$$Z_7\beta \subset Z_6\beta \subset Z_1\beta \subset \breve{D}\beta,\ Z_7\beta \subset Z_6\beta \subset Z_3\beta \subset \breve{D}\beta,$$

i.e., the diagram of the semilattice $V(D,\alpha\circ\beta)$ is a chain if the elements $Z_1\beta$ and $Z_3\beta$ are correlated with set-theoretical inclusions or the diagram of the semilattice $V(D,\alpha\circ\beta)$ has form 7 shown in Fig. 4 if $Z_3\beta \setminus Z_1\beta \neq \varnothing$ and $Z_1\beta \setminus Z_3\beta \neq \varnothing$. This contradicts our proposition that the diagram of the semilattice $V(D,\alpha\circ\beta)$ has form 9 shown in Fig. 4. So, we have $Z_4\beta \neq Z_1\beta$.

g) $Z_3\beta = \breve{D}\beta$ or $Z_1\beta = \breve{D}\beta$. Since five elements of the semilattice $V(D,\alpha\circ\beta)$ are pairwise different subsets of the set X, we have

$$Z_7\beta \subset Z_6\beta \subset Z_1\beta \subset \breve{D}\beta,\ Z_7\beta \subset Z_4\beta \subset Z_1\beta \subset \breve{D}\beta$$

or

$$Z_7\beta \subset Z_6\beta \subset Z_3\beta \subset \breve{D}\beta,\ Z_7\beta \subset Z_4\beta \subset Z_3\beta \subset \breve{D}\beta.$$

i.e., the diagram of the semilattice $V(D,\alpha\circ\beta)$ is a chain if the elements $Z_4\beta$ and $Z_6\beta$ are correlated with set-theoretical enclusions or the diagram of the semilattice $V(D,\alpha\circ\beta)$ has form 6 in Fig. 4 if $Z_6\beta \setminus Z_4\beta \neq \varnothing$ and $Z_4\beta \setminus Z_6\beta \neq \varnothing$. This contradicts our proposition that the diagram of the semilattice $V(D,\alpha\circ\beta)$ has form 9 in Fig. 4. So, we have $Z_3\beta \neq \breve{D}\beta$ and $Z_1\beta \neq \breve{D}\beta$.

h) $Z_6\beta = Z_4\beta$ or $Z_4\beta = Z_3\beta$ or $Z_3\beta = Z_1\beta$. By the definition of the semilattice D we have $Z_6 \cup Z_4 = Z_1$, $Z_4 \cup Z_3 = Z_3 \cup Z_1 = \breve{D}$. From these conditions it follows that $Z_6\beta \cup Z_4\beta = Z_1\beta$ and $Z_4\beta \cup Z_3\beta = Z_3\beta \cup Z_1\beta = \breve{D}\beta$. From this and by the proposition we have $Z_6\beta = Z_4\beta = Z_1\beta$, $Z_4\beta = Z_3\beta = \breve{D}\beta$, $Z_3\beta = Z_1\beta = \breve{D}\beta$. So, $|V(D,\alpha\circ\beta)| \leq 4$. This contradicts our proposition since $|V(D,\alpha\circ\beta)| = 5$. Therefore $Z_6\beta \neq Z_4\beta$ and $Z_4\beta \neq Z_3\beta$ and $Z_3\beta \neq Z_1\beta$.

From the cases $a)-h)$ it follows that the diagram of the semilattice $V(D,\alpha\circ\beta)$ may never have form 9 in Fig. 4. Thus the diagram of the semilattice $V(D,\alpha\circ\beta)$ may be one of diagrams $1-8$ given in Fig. 4. By virtue of Theorems 11.6.1, 11.6.3 and 11.7.2 (See [1]), they are $XI-$semilattices of unions. Therefore, from Theorem 14.20.1 (see [1]) it follows that the binary relation $\alpha\circ\beta$ is a regular element of the semigroup $B_X(D)$.

2) If the diagram of the semilattice $V(D,\beta)$ has form 7 in Fig. 2, then the diagram of any subsemilattice of the semilattice $V(D,\beta)$ may have one of forms $1-7$ of shown in Fig. 5.

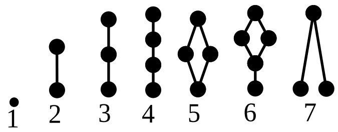

Fig. 5

Since the semilattice $V(D,\alpha\circ\beta)$ has the smallest element, the diagram of the semilattice $V(D,\alpha\circ\beta)$ may have one of forms $1-6$ in Fig. 5. By Theorems 11.6.1 and 11.6.3 (see [1]), they are $XI-$semilattices of unions. Therefore, from Theorem 14.20.1 (see [1]) it follows that the binary relation $\alpha\circ\beta$ is a regular element of the semigroup $B_X(D)$.

3) The diagram of the semilattice $V(D,\beta)$ has form 6 shown in Fig. 2. Then the diagram of any subsemilattice of the semilattice $V(D,\beta)$ may have one of forms $1-8$ shown in Fig. 2.

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 4, July 2015

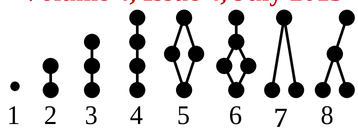

Fig. 6

Since the semilattice $V(D,\alpha\circ\beta)$ has the smallest element, the diagram of the semilattice $V(D,\alpha\circ\beta)$ may have one of forms $1-6$ given in Fig. 6. By Theorems 11.6.1 and 11.6.3 (see [1]) they are $XI-$semilattices of unions. Therefore, from the Theorem 14.20.1 (See [1]) it follows that the binary relation $\alpha\circ\beta$ is a regular element of the semigroup $B_X(D)$.

$4)$ The diagram of the semilattice $V(D,\beta)$ has form 5 given in Fig. 2. Then the diagram of any subsemilattice of the semilattice $V(D,\beta)$ may have one of forms $1-5$ shown in Fig. 2.

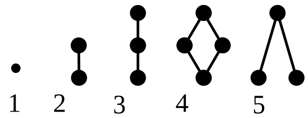

Fig. 7

Since the semilattice $V(D,\alpha\circ\beta)$ has the smallest element, the diagram of the semilattice $V(D,\alpha\circ\beta)$ may have one of forms $1-4$ shown in Fig. 7. By Theorems 11.6.1 and 11.6.3 (see [1]) it follows that they are $XI-$semilattices of unions. Therefore, from the Theorem 14.20.1 (see [1]) it follows that the binary relation $\alpha\circ\beta$ is a regular element of the semigroup $B_X(D)$.

$5)$ the diagram of the semilattice $V(D,\beta)$ has forms $1-4$ given in Fig. 2. Then the diagram of any subsemilattice of the semilattice $V(D,\beta)$ is a finite chain. By Theorem 11.6.1 (see [1]) it follows that they are $XI-$semilattices of unions. Therefore, from Theorem 14.20.1 (see [1]) it follows that the binary relation $\alpha\circ\beta$ is a regular element of the semi group $B_X(D)$.

The theorem is proved.

REFERENCES

[1] Ya. Diasamidze, Sh. Makharadze. Complete semi groups of binary relations. Monograph. Kriter, Turkey, 2013, 520 p.

[2] Ya. Diasamidze, Sh. Makharadze. Complete semi groups of binary relations. Monograph. M., Sputnik+, 2010, 657 p. (Russian).

[3] Ya. I. Diasamidze. Complete semi groups of binary relations. Journal of Mathematical Sciences, Plenum Publ. Corp., New York, vol. 117, No. 4, 2003, 4271-4319.

[4] Diasamidze Ya., Makharadze Sh., Partenadze G., Givradze O. On finite $X-$semilattices of unions. Journal of Mathematical Sciences, Plenum Publ. Corp., New York, 141, No. 4, 2007, 1134-1181.

[5] Diasamidze Ya., Makharadze Sh., Maximal subgroups of complete semi groups of binary relations. Proc. A. Razmadze Math. Inst. 131, 2003, 21-38.

[6] Diasamidze Ya., Makharadze Sh., Diasamidze Il., Idempotents and regular elements of complete semi groups of binary relations. Journal of Mathematical Sciences, Plenum Publ. Corp., New York, 153, No. 4, 2008, 481-499.

[7] Diasamidze Ya., Makharadze Sh., Rokva N., On $XI-$semilattices of unions. Bull. Georg. Nation. Acad. Sci., 2, No. 1, 2008, 16-24.

[8] Diasamidze Ya., Bakuridze A. Idempotent elements of complete semi groups of binary relations defined by semilattices of the class $\Sigma_4(X,8)$ (to appear).

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 4, July 2015

[9] Diasamidze Ya., Bakuridze A. Regular elements of complete semi groups of binary relations defined by semilattices of the class $\Sigma_4(X,8)$ (to appear).