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On the Use of Exact Coefficients of the GEV Distribution Matrix Variance-Covariance for the Maxima and Minima

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***Abstract*— The method of maximum likelihood has been consistently better than the other alternatives for the estimation of parameters and quantiles of the GVE distribution for the maxima and minima. In these processes a key feature is that of using the variance-covariance coefficients, most of the times taking into account those that have been obtained by numerical integration and more seldom those that came from a close solution of the integrals that are related with such procedure. In the paper, are presented both approaches and with the aid of two examples in each case, a wider application of the numerical integration approach is observed when using actual extreme value data given that the close solution is kind of restrictive and only applies to a small interval of values of the shape parameter of the GEV distribution..**

***Index Terms*—exact evaluation of coefficients, general extreme value distribution, maximum likelihood, numerical integration evaluation of coefficients, variance-covariance matrix.**

I. INTRODUCTION

The method of maximum likelihood (ML) has been acknowledged as one of the best methods for parameter and quantile estimation of probability distribution functions. The properties of its estimators like the invariance, the asymptotically unbiasness, sufficiency, consistency, and efficiency, [1] and [2], and the remarkable suitability when being applied to cumbersome mathematical expressions in its likelihoods functions, have gained it the well-known condition of prime choice for solving problems of parameter and quantile estimation of probability distribution functions. This is the case for the GEV distribution function for both the maxima and the minima.

The use of the general extreme value (GEV) distribution function, [3] and [4], for flood and low flow frequency analyses had been used now for a number of years and several estimators for the parameters and quantiles have been proposed to achieve such tasks. A good example of that for the case of flood frequency analysis, using the GEV distribution, is [5]. When the low flow case is explored, examples of estimation of parameters and quantiles for the GEV distribution for the minima is [6].

It is the purpose of this paper to compare the approaches of exact evaluation of the coefficients of the variance-covariance matrix of the parameters of the GEV distribution for the cases of the maxima and minima with the numerical evaluation of such coefficients and see if there are significant discrepancies in both methods.

II. THE GENERAL EXTREME VALUE DISTRIBUTION

The GEV distribution function for the maxima is, [5]:

$$F(x) = \exp\left\{-\left[1-\frac{\beta(x-x_0)}{\alpha}\right]^{1/\beta}\right\} \tag{1}$$

where $F(x)$ is the GEV distribution function for the maxima, x_0 is the location parameter, α is the scale parameter and β is the shape parameter. And its corresponding density distribution function is:

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$$f(x) = \frac{1}{\alpha}\left[1 - \frac{\beta(x - x_0)}{\alpha}\right]^{1/\beta - 1} \exp\left\{-\left[1 - \frac{\beta(x - x_0)}{\alpha}\right]^{1/\beta}\right\}$$

(2)

The GEV distribution function for the minima is, [6]:

$$\Pi(x) = \exp\left\{-\left[1 - \frac{\beta(\omega - x)}{\alpha}\right]^{1/\beta}\right\} \quad (3)$$

where *Π(x)* is the GEV distribution function for the minima, ω is the location parameter, α is the scale parameter and β is the shape parameter. And its corresponding density distribution function is:

$$\pi(x) = \frac{1}{\alpha}\left[1 - \frac{\beta(\omega - x)}{\alpha}\right]^{1/\beta - 1} \exp\left\{-\left[1 - \frac{\beta(\omega - x)}{\alpha}\right]^{1/\beta}\right\} \quad (4)$$

III. THE METHOD OF MAXIMUM LIKELIHOOD

The likelihood function for N independent and identically distributed random variables X_1, X_2, … ,Xn can be obtained as the joint probability density function, that is, [1]:

$$L(x,\theta) = \prod_{i=1}^{N} f(x_i) \quad (5)$$

where θ is the parameter vector and f (.) is the probability density function.
The logarithmic version of the former equation is:

$$LnL(x,\theta) = \sum_{i=1}^{N} Lnf(x) \quad (6)$$

Based in the statements of the previous section, the logarithmic likelihood function of the GEV distribution function for the maxima is, [3]:

$$Ln\ L(x; x_0, \alpha, \beta) = -NLn(\alpha) \ -\sum_{i=1}^{N}\left[1 - \frac{\beta(x - x_0)}{\alpha}\right]^{1/\beta} + \left(\frac{1}{\beta} - 1\right)\sum_{i=1}^{N} Ln\left[1 - \frac{\beta(x - x_0)}{\alpha}\right] \quad (7)$$

where *Ln L(.)* is the logarithm of the likelihood function.
And the corresponding logarithmic likelihood function of the GEV distribution function for the minima is, [5]:

$$LnL(x; x_0, \alpha, \beta) = -NLn(\alpha) \ -\sum_{i=1}^{N}\left[1 - \frac{\beta(\omega - x)}{\alpha}\right]^{1/\beta} + \left(\frac{1}{\beta} - 1\right)\sum_{i=1}^{N} Ln\left[1 - \frac{\beta(\omega - x)}{\alpha}\right] \quad (8)$$

IV. THE VARIANCE-COVARIANCE MATRIX

The variance-covariance for the GEV distribution function for the maxima is, [5]:

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$$[V] = \begin{bmatrix} Var(x_0) & Cov(x_0,\alpha) & Cov(x_0,\beta) \\ Cov(\alpha,x_0) & Var(\alpha) & Cov(\alpha,\beta) \\ Cov(\beta,x_0) & Cov(\beta,\alpha) & Var(\beta) \end{bmatrix} = \begin{bmatrix} E\left(-\frac{\partial^2 LL}{\partial x_0^{\,2}}\right) & E\left(-\frac{\partial^2 LL}{\partial x_0 \partial \alpha}\right) & E\left(-\frac{\partial^2 LL}{\partial x_0 \partial \beta}\right) \\ E\left(-\frac{\partial^2 LL}{\partial \alpha \partial x_0}\right) & E\left(-\frac{\partial^2 LL}{\partial\ \alpha^2}\right) & E\left(-\frac{\partial^2 LL}{\partial \alpha \partial \beta}\right) \\ E\left(-\frac{\partial^2 LL}{\partial \beta \partial x_0}\right) & E\left(-\frac{\partial^2 LL}{\partial \beta \partial \alpha}\right) & E\left(-\frac{\partial^2 LL}{\partial \beta^2}\right) \end{bmatrix}^{-1} \quad (9)$$

and equation (9) can be expressed in the following form:

$$[V] = \begin{bmatrix} Var(x_0) & Cov(x_0,\alpha) & Cov(x_0,\beta) \\ Cov(\alpha,x_0) & Var(\alpha) & Cov(\alpha,\beta) \\ Cov(\beta,x_0) & Cov(\beta,\alpha) & Var(\beta) \end{bmatrix} = \frac{1}{N}\begin{bmatrix} \alpha^2 b & \alpha^2 h & \alpha f \\ \alpha^2 h & \alpha^2 a & \alpha g \\ \alpha f & \alpha g & c \end{bmatrix} \quad (10)$$

Where a, b, c, f, g and h are the coefficients of the variance-covariance matrix for the GEV distribution for the maxima.

The variance-covariance for the GEV distribution function for the minima is, [5]:

$$[V] = \begin{bmatrix} Var(\omega) & Cov(\omega,\alpha) & Cov(\omega,\beta) \\ Cov(\alpha,\omega) & Var(\alpha) & Cov(\alpha,\beta) \\ Cov(\beta,\omega) & Cov(\beta,\alpha) & Var(\beta) \end{bmatrix} = \begin{bmatrix} E\left(-\frac{\partial^2 LL}{\partial \omega^2}\right) & E\left(-\frac{\partial^2 LL}{\partial \omega \partial \alpha}\right) & E\left(-\frac{\partial^2 LL}{\partial \omega \partial \beta}\right) \\ E\left(-\frac{\partial^2 LL}{\partial \alpha \partial \omega}\right) & E\left(-\frac{\partial^2 LL}{\partial\ \alpha^2}\right) & E\left(-\frac{\partial^2 LL}{\partial \alpha \partial \beta}\right) \\ E\left(-\frac{\partial^2 LL}{\partial \beta \partial \omega}\right) & E\left(-\frac{\partial^2 LL}{\partial \beta \partial \alpha}\right) & E\left(-\frac{\partial^2 LL}{\partial \beta^2}\right) \end{bmatrix}^{-1} \quad (11)$$

and equation (11) can be expressed as:

$$[V] = \begin{bmatrix} Var(\omega) & Cov(\omega,\alpha) & Cov(\omega,\beta) \\ Cov(\alpha,\omega) & Var(\alpha) & Cov(\alpha,\beta) \\ Cov(\beta,\omega) & Cov(\beta,\alpha) & Var(\beta) \end{bmatrix} = \frac{1}{N}\begin{bmatrix} \alpha^2 b & \alpha^2 h & \alpha f \\ \alpha^2 h & \alpha^2 a & \alpha g \\ \alpha f & \alpha g & c \end{bmatrix} \quad (12)$$

Where a, b, c, f, g and h are the coefficients of the variance-covariance matrix for the GEV distribution for the minima.

V. EXACT VARIANCE-COVARIANCE MATRIX COEFFICIENTS OF THE GEV DISTRIBUTION FOR THE MAXIMA AND MINIMA

a) For the maxima

The expected values inside the Fisher's information matrix, equation (9), for the GEV distribution for the maxima, have been obtained by [7] for the interval -0.50 < β < 0.50, as:

$$E\left(-\frac{\partial^2 LL}{\partial x_0^{\,2}}\right) = \frac{N}{\alpha^2} p \quad (13)$$

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$$E\left(-\frac{\partial^2 LL}{\partial \alpha^2}\right) = \frac{N}{\alpha^2\beta^2}[1-2(1-\beta)\Gamma(1-\beta)+p]$$
(14)

$$E\left(-\frac{\partial^2 LL}{\partial \beta^2}\right) = \frac{N}{\beta^2}\left[\frac{\pi^2}{6}+\left(1-\varepsilon-\frac{1}{\beta}\right)^2+\frac{2q}{\beta}+\frac{p}{\beta^2}\right]$$
(15)

$$E\left(-\frac{\partial^2 LL}{\partial x_0 \partial \alpha}\right) = \frac{N}{\alpha^2\beta}[p-(1-\beta)\Gamma(1-\beta)] \quad (16)$$

$$E\left(-\frac{\partial^2 LL}{\partial x_0 \partial \beta}\right) = \frac{N}{\alpha\beta}\left[-\frac{p}{\beta}-q\right] \quad (17)$$

$$E\left(-\frac{\partial^2 LL}{\partial \alpha \partial \beta}\right) = \frac{N}{\alpha\beta^2}\left[1-\varepsilon-\frac{[1-(1-\beta)\Gamma(1-\beta)]}{\beta}-\frac{p}{\beta}-q\right]$$
(18)

where:

$$p = (1-\beta)^2\Gamma(1-2\beta)$$
(19)

$$q = (1-\beta)\Gamma(1-\beta)\left[\psi(1-\beta)-\frac{(1-\beta)}{\beta}\right] \quad (20)$$

and Γ(.) is the complete Gamma function, ψ(.) is the Digamma function and ε is the Euler's constant (equal to 0.5772157).
In table I, some values of the exact coefficients of the variance-covariance matrix of the GEV distribution for the maxima are shown.

TABLE I. EXACT COEFFICIENTS OF THE VARIANCE-COVARIANCE MATRIX OF THE GEV DISTRIBUTION FOR THE MAXIMA

β	a	b	c	f	g	h
-0.4	0.553	2.454	3.017	2.343	-0.168	0.088
-0.3	0.576	1.538	1.397	1.050	-0.021	0.207
-0.2	0.601	1.311	0.886	0.627	0.046	0.257
-0.1	0.623	1.241	0.621	0.397	0.096	0.291
0.0	0.650	1.25	0.48	0.26	0.15	0.34
0.1	0.689	1.300	0.376	0.175	0.195	0.391
0.2	0.791	1.444	0.307	0.155	0.257	0.557
0.3	1.588	2.641	0.315	0.463	0.525	1.633

b) For the minima
The expected values inside the Fisher's information matrix, equation (11), for the GEV distribution for the minima, have been obtained, for the interval -0.50 < β < 0.50, as:

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$$E\left(-\frac{\partial^2 LL}{\partial \omega^2}\right) = \frac{N}{\alpha^2} p$$

(21)

$$E\left(-\frac{\partial^2 LL}{\partial \alpha^2}\right) = \frac{N}{\alpha^2 \beta^2}\left[1 - 2(1-\beta)\Gamma(1-\beta) + p\right]$$

(22)

$$E\left(-\frac{\partial^2 LL}{\partial \beta^2}\right) = \frac{N}{\beta^2}\left[\frac{\pi^2}{6} + \left(1 - \varepsilon - \frac{1}{\beta}\right)^2 + \frac{2q}{\beta} + \frac{p}{\beta^2}\right]$$

(23)

$$E\left(-\frac{\partial^2 LL}{\partial \omega \partial \alpha}\right) = \frac{N}{\alpha^2 \beta}\left[(1-\beta)\Gamma(1-\beta) - p\right]$$

(24)

$$E\left(-\frac{\partial^2 LL}{\partial \omega \partial \beta}\right) = \frac{N}{\alpha \beta}\left[\frac{p}{\beta} + q\right] \quad (25)$$

$$E\left(-\frac{\partial^2 LL}{\partial \alpha \partial \beta}\right) = \frac{N}{\alpha \beta^2}\left[1 - \varepsilon - \frac{\left[1 - (1-\beta)\Gamma(1-\beta)\right]}{\beta} - \frac{p}{\beta} - q\right] \quad (26)$$

where:

$$p = (1-\beta)^2 \Gamma(1-2\beta) \quad (27)$$

$$q = (1-\beta)\Gamma(1-\beta)\left[\psi(1-\beta) - \frac{(1-\beta)}{\beta}\right]$$

(28)

and $\Gamma(.)$ is the complete Gamma function, $\psi(.)$ is the Digamma function and ε is the Euler's constant (equal to 0.5772157).

In table II, a few values of the exact coefficients of the variance-covariance matrix of the GEV distribution for the minima are shown.

TABLE II. EXACT COEFFICIENTS OF THE VARIANCE-COVARIANCE MATRIX OF THE GEV DISTRIBUTION FOR THE MINIIMA

β	a	b	c	f	g	h
0.0	0.772	1.079	0.546	-0.208	0.285	-0.330
0.1	0.608	1.227	0.400	-0.242	0.185	-0.216
0.2	0.584	1.202	0.330	-0.220	0.214	-0.092
0.3	0.580	1.173	0.265	-0.193	0.233	0.035
0.4	0.595	1.141	0.206	-0.162	0.242	0.164
0.45	0.609	1.125	0.178	-0.146	0.242	0.231

In order to observe the differences between the results provided by the exact coefficients and those obtained by using numerical integration, for both the maxima and the minima, two different examples had been constructed to provide information on how different results may be expected when using one or the other approach.

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VI. NUMERICAL INTEGRATION VARIANCE-COVARIANCE MATRIX COEFFICIENTS OF THE GEV DISTRIBUTION FOR THE MAXIMA AND MINIMA

The coefficients of the variance-covariance of the GEV distribution for the maxima and the minima may be obtained through numerical integration, too. So, the values of the coefficients of the GEV distribution for the maxima had been obtained by [6], are shown in table III.

TABLE III. COEFFICIENTS OF THE VARIANCE-COVARIANCE MATRIX OF THE GEV DISTRIBUTION FOR THE MAXIMA (NUMERICAL INTEGRATION)

β	a	b	c	f	g	h
-0.4	1.043	1.291	0.834	0.264	-0.088	0.798
-0.3	0.914	1.286	0.736	0.271	-0.018	0.685
-0.2	0.806	1.278	0.642	0.272	0.044	0.571
-0.1	0.718	1.261	0.542	0.260	0.096	0.453
0.0	0.650	1.250	0.480	0.260	0.150	0.340
0.1	0.608	1.227	0.402	0.242	0.185	0.216
0.2	0.584	1.203	0.335	0.223	0.215	0.093
0.3	0.581	1.176	0.276	0.199	0.237	-0.033
0.4	0.598	1.150	0.235	0.177	0.253	-0.159
0.5	0.639	1.128	0.222	0.165	0.267	-0.284
0.6	0.706	1.113	0.249	0.167	0.285	-0.404
0.7	0.803	1.100	0.309	0.176	0.309	-0.523
0.8	.946	1.073	0.382	0.167	0.341	-0.651
0.9	1.229	1.012	0.453	0.074	0.453	-0.829
1.0	1.274	1.002	0.007	-0.004	0.096	-1.055

The corresponding values of the coefficients of the variance-covariance of the GEV distribution for the minima, [6], are sown in table IV.

TABLE IV. COEFFICIENTS OF THE VARIANCE-COVARIANCE MATRIX OF THE GEV DISTRIBUTION FOR THE MINIIMA (NUMERICAL INTEGRATION)

β	a	b	c	f	g	h
0.0	0.772	1.079	0.546	-0.208	0.285	-0.330
0.1	0.637	1.241	0.491	-0.273	0.217	-0.222
0.2	0.613	1.224	0.449	-0.269	0.253	-0.100
0.3	0.602	1.203	0.404	-0.252	0.270	0.031
0.4	0.608	1.184	0.394	-0.256	0.289	0.141
0.5	0.670	1.111	0.394	-0.225	0.323	0.258
0.6	0.993	1.110	.816	-0.403	0.677	0.183

VII. NUMERICAL EXAMPLES

a) For the maxima

The annual maximum discharges of the St. Mary's River at Stillwater, Nova Scotia, Canada, [8], in the period 1915-1986, were used to test the possible differences in the results provided by the exact coefficients and those obtained by numerical integration. The results are shown in tables V and VI. A graphical display of these results is shown in figure 1.

TABLE V. VALUES OF THE PARAMETERS OF THE GEV DISTRIBUTION FOR THE MAXIMA

Method	Parameters		
	X_0	α	β
Exact coefficients approach	348.81	107.74	-0.015
Numerical integration approach	347.46	106.92	-0.036

TABLE VI. VALUES OF THE DESIGN VALUES FOR DIFFERENT RETURN PERIODS COMPUTED BY USING THE GEV DISTRIBUTION FOR THE MAXIMA

Method	Design Values (m^3/s)					
	Q_2	Q_5	Q_{10}	Q_{20}	Q_{50}	Q_{100}

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Exact coefficients approach	386	512	598	683	796	883
Numerical integration approach	388	512	595	676	782	862

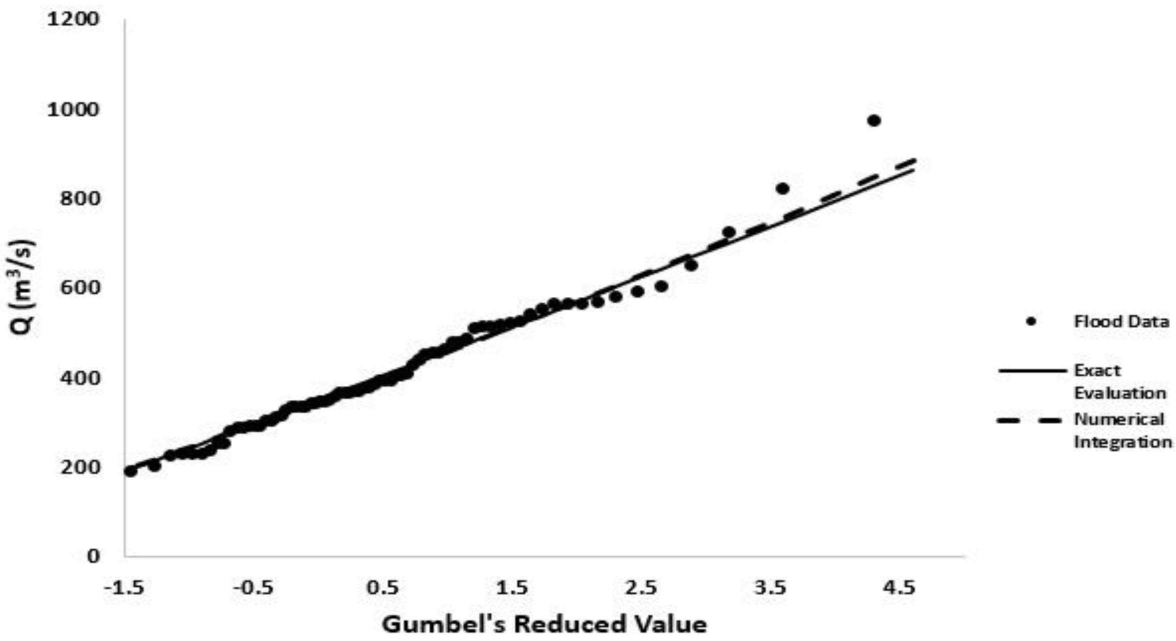

Fig. 1 Empirical and theoretical frequency curves for gauging station St. Mary's River at Stillwater, Nova Scotia, Canada, (1915-1986).

a) For the minima

The annual one-day low-flows discharges of the gauging station Villalba, Mexico, [9], in the period 1939-1991, were used to test the possible differences in the results provided by the exact coefficients and those obtained by numerical integration. The results are shown in tables VII and VIII. A graphical display of these results is shown in figure 2.

TABLE VII. VALUES OF THE PARAMETERS OF THE GEV DISTRIBUTION FOR THE MINIMA

Method	Parameters		
	ω	α	β
Exact coefficients	0.37	0.17	0.53
Numerical integration approach	0.38	0.13	0.43

TABLE VIII. VALUES OF THE DESIGN VALUES FOR DIFFERENT RETURN PERIODS COMPUTED BY USING THE GEV DISTRIBUTION FOR THE MINIMA

Method	Design Values (m^3/s)					
	Q_2	Q_5	Q_{10}	Q_{20}	Q_{50}	Q_{100}
Exact coefficients approach	0.31	0.20	0.16	0.13	0.10	0.09
Numerical integration approach	0.33	0.23	0.19	0.16	0.13	0.11

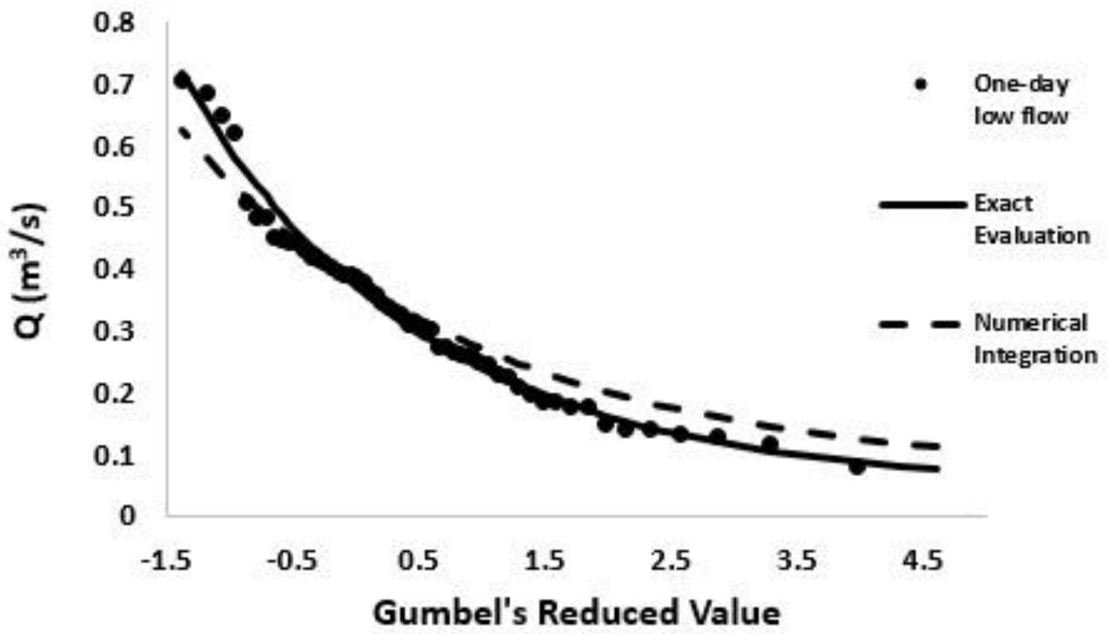

Fig. 2 Empirical and theoretical frequency curves for gauging station Villalba, Mexico (1939-1991).

VIII. CONCLUSIONS

The coefficients of the variance-covariance matrix for the GEV distribution, for the maxima and minima, have been presented when they are evaluated through the exact solution and by numerical integration.

The differences in the final results, when they have been applied to a set of data of maximum annual discharges and one-day low flow data, in the examples that are shown in the paper, showed very little differences, what it may be indicative that both approaches are almost the same.

In the case of the maximum annual discharges, the results of the design values for different return periods, are slightly underestimated when the numerical approach is used.

In the case of the one-day low flow data, the results of the design values for different return periods, are consistently overestimated when the numerical approach is used.

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