ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 3, May 2015

Funnel Vortices and Fluid Particles Interaction

José Roberto Mercado-Escalante[1], Pedro Antonio Guido-Aldana[1], Gilberto Zetina-Domínguez[2]
[1]Mexican Institute of Water Technology, [2]Federal Electricity Commission

***Abstract*— The purpose of this paper is to obtain an expression for the fluid force on a particle in suspension enabling their application to sediment transport in a channel. The movement of a fluid is described by the fractional Navier-Stokes equation that relates the variations of velocity and pressure as the primary variables. The need for objectivity forced to pass to the vorticity, whereas the conservation of mass allows a potential way to velocity through the stream function, so the equation is contained in non-primary variables. The problem of the so-called free vortex is considered which provides an expression for the velocity, by performing the reverse curl. In the approximation of the boundary layer, the stream function through the Biot-Savart operator is displayed. An expression is obtained for the stress, and one for the friction force on the boundary. Coherent structures near the wall are modeled as vortex knots. Experimental result of the angular velocity of vortex funnel is resumed, and the winding number of the vortex knot is formulated. A representation of the force on a particle is finally obtained.**

***Index Terms*— Fractional Navier-Stokes equation, funnel vortices, sediment transport, solid particles.**

I. INTRODUCTION

The law that governs the incompressible fluid motion is known as Navier-Stokes fractional, along with the conservation of mass or no divergence, [1] - [2].

According to our description of the movement of fluids, essentially relies on the momentum flow generated by viscous force between adjacent layers with different relative velocities. But this phenomenon cannot be described by a local operator due to the involvement of a frictional force, so it must be expressed by a non-local operator. There is a Darcy fractional law that states that Darcy flow is proportional and opposite of the fractional gradient of momentum, per unit volume. This Darcy flow generates a momentum change, so that according to Newton's law the change rate of momentum is the negative divergence of Darcy flow, with opposite sign. Then we can consider changes in fluid pressure being that pressure gradient also contributes to the change in momentum. Next, presence of a body force is taken into account, like the gravitational field, which also participates in the momentum balance. Finally, change in momentum for incompressible fluid results in a velocity change that contains the local and advective acceleration.

Momentum per unit volume is represented by $\rho\mathbf{u}$, the fractional gradient is expressed by $\nabla_M^\beta \rho\mathbf{u}$, momentum diffusivity is the α-kinematic viscosity, ν_α, $\alpha = 1+\beta$, β is the index of spatial occupation, M the mixing scale for the different spatial directions, and so Darcy flow of momentum is:

$$\mathbf{q}_D = -\nu_\alpha \nabla_M^\beta \rho\mathbf{u} \tag{1}$$

We denote (p, ϕ, ρ) as pressure, body force potential and fluid density; finally, we obtain (2), it is achieved choosing M such that the flow is proportional to the negative fractional Laplacian. And consequently it can be said that the vorticity energize the evolution of the velocity field against viscosity under the restriction of energy conservation, [1], [3]:

$$\frac{\partial}{\partial t}\mathbf{u} = -\nu_\alpha(-\Delta)^{\alpha/2}\mathbf{u} + \mathbf{u}\times rot\mathbf{u} - \nabla\left(\frac{1}{2}(\mathbf{u}\cdot\mathbf{u}) + \frac{p}{\rho} + \phi\right) \tag{2}$$

Boundary layer equations arise from fractional Navier-Stokes equation as an approximation induced by the premise of its relatively thin thickness, and is summarized as follows: the main direction for velocity is the downstream and a large vertical velocity gradient, compared with the longitudinal gradient, leading to the velocity fulfilling the condition of no slippage in the wall, and on the contrary, pressure gradients are light in the vertical direction compared with the longitudinal strengths, [4].

Dimensionless variables are obtained from momentum equation and non-divergence, by invariance structure and identifying the length and velocity characteristics. The relation of re-normalization for coordinates and velocities is as in (3), being $\left(l,\ \ U,\ \ R_{e\beta}=Ul^{\beta}/\nu_{\alpha}\right)$, characteristic length, external velocity, indexed Reynolds number. And we can define them as follows:

$$x=\frac{x}{l},\quad y=\left(R_{e\beta}\right)^{1/\alpha}\frac{y}{l},\quad u=\frac{u}{U},\quad v=\left(R_{e\beta}\right)^{1/\alpha}\frac{v}{U} \tag{3}$$

Stationary version of boundary layer equations is written (4) and it is observed that when the advective force, which is driven by the motor of the vorticity, exceeds the viscous force, we will have: an adverse pressure gradient and consequently the emergence of the vortices.

$$(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu_{\alpha}\nabla_{M}^{\alpha}\mathbf{u}=-\nabla p/\rho \tag{4}$$

II. SEQUENCE OF VORTICITY

Incompressible nature of the fluid is manifested in the form of mass conservation, which is expressed by $div\mathbf{u}=0$, (solenoidal $\mathbf{u}$), so that the velocity is recovering from a potential vector, in a simply connected domain:

$$\mathbf{u}=rotA \tag{5}$$

And because the identity $divrot\,(\cdot)=0$, the vorticity is given by $\boldsymbol{\omega}=rot\,(rotA)$, so the Poisson equation arises $\nabla^{2}A=-\boldsymbol{\omega}(r)$. The solution of this equation is sought by the method of Green's function G without boundary and to the point source, and then extend it by convolution, so is $A=G*\boldsymbol{\omega}=\frac{1}{4\pi}\int dV\,\frac{\boldsymbol{\omega}(x')}{|x-x'|}$. And it can be seen that the Laplacian of the vector potential is the vorticity, with the sign changed. Therefore, $B=-\frac{1}{4\pi}\frac{x}{|x|^{3}}\times$, the Biot-Savart operator provides rotational inversion, and the velocity is expressed by, [5]:

$$\mathbf{u}(x)=(B*_{x'}\boldsymbol{\omega})(x) \tag{6}$$

With the representation obtained at (6), we create a succession and with it, their linear combinations, and, finally, we present a new sequence (7) and expect that it converge to a solution,

$$u_{k}(x)=\left(C*_{i}\left(B*_{x'}\omega_{k}\right)(x)\right)_{k}\xrightarrow{k}u(x) \tag{7}$$

Temporarily we suppress the subscript of the succession in the vorticity. We introduce the hypotheses that describe vorticity through a vortex knot, (torus knot),

$$\omega_{3}=\frac{1}{4}q^{a}\sin(qt+\frac{\pi}{2}a) \tag{8}$$

The other two components are, [6], [7]:

$$\omega_{2}=p^{a}\sin\left(pt+\tfrac{\pi}{2}a\right)+\tfrac{(p+q)^{a}}{8}\sin\left((p+q)t+\tfrac{\pi}{2}a\right)+\tfrac{(p-q)^{a}}{8}\sin\left((p-q)t+\tfrac{\pi}{2}a\right) \tag{9}$$

$$\omega_{1}=p^{a}\cos\left(pt+\tfrac{\pi}{2}a\right)+\tfrac{(p+q)^{a}}{8}\cos\left((p+q)t+\tfrac{\pi}{2}a\right)+\tfrac{(p-q)^{a}}{8}\cos\left((p-q)t+\tfrac{\pi}{2}a\right) \tag{10}$$

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 3, May 2015

The torus knot is the Cartesian product of two circles: one equatorial, horizontal and large, and the other, small and vertical. With fewer turns around the equatorial, p, and more turns around the vertical, q; being p y q coprime, $2 \le p < q$, with winding number $w = q / p$. The order of the fractional derivative is a, so we re-scale the knot when a grows, and vice versa.

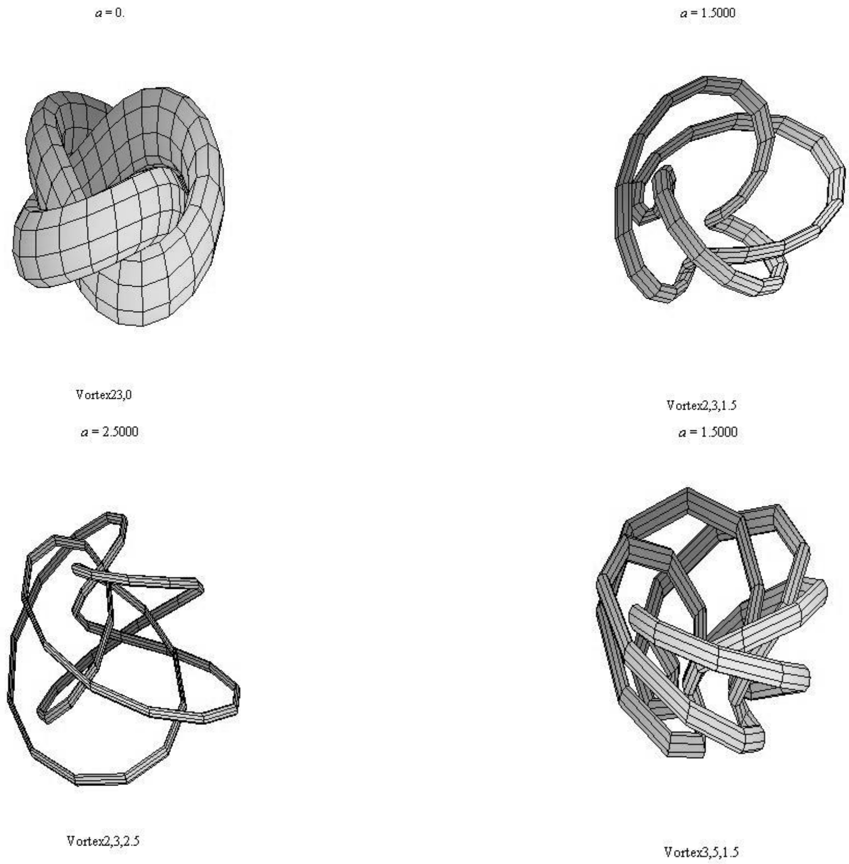

Fig. 1. Knot representation considering different values of the order of the fractional derivative.

In Fig. 1, we show the vortex known as trefoil knot, for three values of the order of the derivative, and also show the knot vortex 3,5 with the order of the derivative of 1.5, to compare with the Vortex 2,3 and the same order of derivative.

III. VORTICITY AND BOUNDARY LAYER

In the approximation of the boundary layer, we want to identify the most important contribution vorticity. Being $\vec{u} \times (\nabla \times \vec{u})$ the term associated to vorticity, with $i\left(v(\partial_x v - \partial_y u) - w(\partial_z u - \partial_x w)\right)$ the first component, that reduced to $iv(\partial_x v - \partial_y u)$. Moreover, the contribution of the kinetic energy gradient $-\nabla\left(\frac{1}{2}\vec{u} \cdot \vec{u}\right)$ is reduced to $-\frac{1}{2}\nabla(u^2) = -u\,\partial_x u$. Therefore the two longitudinal contributions to the acceleration, by the vorticity and the quadratic, produce $i\left(v(\partial_x v - \partial_y u) - u\,\partial_x u\right) \approx -i(u\,\partial_x, v\partial_y)u \approx -i(\vec{u} \cdot \nabla)\vec{u}$. Hence the approximation is:

$$-\left((\vec{u} \cdot \nabla)\vec{u}\right)_i \approx \left(\vec{u} \times (\nabla \times \vec{u})\right)_i - \left(\nabla\left(\frac{1}{2}\vec{u} \cdot \vec{u}\right)\right)_i + \varepsilon_i \qquad (11)$$

Consequently, in the boundary layer approximation, primarily are interested in third component of vorticity, which points toward binormal.

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 3, May 2015

IV. POTENTIAL

Potential form of velocity is assumed though the stream function; for the longitudinal component is given by $u=\frac{\partial}{\partial y}\psi$, this suggests that it should be $u=\frac{\partial}{\partial y}(G*\omega_3)$, so is inferred that the stream function supports the representation $\psi=G*\omega_3+\varepsilon$. Moreover, for the vertical velocity component we have $v=-\frac{\partial}{\partial x}\psi$, which also induces $\psi=G*\omega_3+\varepsilon$, thus one has $\psi=G*\omega_3+\varepsilon$, as the approximation of the stream function.

V. STRESS

In the approximation of Blasius, main velocity is dimensionless by external velocity and the result is represented by the derivative of the sub-potential function with respect to its argument, which is the variable of similarity, [8]:

$$\eta=y/\lambda_{x\beta},\quad \lambda_{x\beta}=\left(\frac{v_\alpha}{U}x\right)^{1/\alpha},\quad \alpha=1+\beta,\quad \frac{u}{U}=g'(\eta) \tag{12}$$

Since in principle stress can be calculated by $\rho v_\alpha \partial_y^\beta \frac{\partial}{\partial y}\psi(u,v)$, and the stream function is approximated by $\psi=G*\omega_3=\frac{r_t^2}{4\pi}\int dl(x')\frac{\omega_3}{|x-x'|}$, the transverse gradient of the stream function is calculated, and the stress is represented by (13):

$$\tau_{xy}=\rho v_\alpha \frac{r_t^2}{4\pi}\int dl(x')\partial_y^\beta\left(\frac{y-y'}{r^3}\right)(-\omega_3) \tag{13}$$

For the frictional force $F_f/l_3=2\int_0^l \tau_{xy}dx$ per unit length binormal, integrated over the longitudinal coordinate and (14) is obtained,

$$F_f/l_3=\rho v_\alpha r_t^2\cdot\frac{1}{2\pi}\int dl(x')(-\omega_3)\cdot\int_0^l \partial_y^\beta\left(\frac{y-y'}{r^3}\right)dx \tag{14}$$

According to [9], the angular velocity of funnel vortex is estimated by $\varpi=2\pi\frac{u(y)}{\lambda(y)}$, being (u,λ) the main velocity and the spatial period of vortex, respectively. Moreover, our hypothesis is to describe the vorticity tubes through vortex knots, which have as main characteristic the winding number $w=q/p$, so we postulate there is a relationship between ϖ and w.

Reference [10] provides that the adverse pressure gradient is the origin of the vortices. The vorticity field is tangential to the surface to high Reynolds numbers and a second vortex is caused by the adverse pressure gradient inducing by primary vortex, which begins with a rash on needle within a narrow horizontal band. Therefore, we assume that the vortex is formed from an adverse pressure gradient that begins with the formation of the needle into the narrow horizontal band near the boundary surface.

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 3, May 2015

In the expression of the force (14), we approximate the factor $\frac{1}{2\pi}\int dl(x')(-\omega_3)$ for the $\frac{1}{2}\int dl(x')\left(-\frac{\Delta w^{a-1}}{4}\frac{\sin\left(\Delta wt'+\frac{\pi}{2}a\right)}{\pi\Delta w}\right)$, under the concept of a succession of Dirac and being $\Delta w = w - \varpi$. So in the limit is $\frac{1}{4}\operatorname{Im}\left(e^{i\left(\frac{a}{2}+1\right)\pi}\right)\frac{1}{2\pi}\int dl(x')(\Delta w^{a-1})e^{i\Delta wt'}$, which contributes to the force, by a factor of the following type $\frac{1}{4}\rho\nu_\alpha r_t^2\cdot(\varpi)^{a-1}$.

For other factor, $\int_0^l \partial_y^\beta\left(\frac{y-y'}{r^3}\right)dx$ the chain rule is used for the first derivative and $\left(\frac{U}{\nu_\alpha}\right)^{1/\alpha}\int_0^l x^{-1/\alpha}\partial_y^{\beta-1}(F'(\eta))dx$ is the result. Finally, for the force we can draw the factor (15),

$$k_1\frac{(U)^{a-1+1/\alpha}}{(\lambda(y))^{a-1}} \tag{15}$$

In the approximation of Saint-Venant, if we imagine the force as a body force we can compare with the results presented in [11], which concluded that the force has an invariance property of the form, so the Lie derivative is proportional to the force itself, being the variables the velocity and size of the hydraulic head. We display the force by $\frac{(U)^A}{(y)^B}$, comparing this expression with (15) and taking into account the experimental results, in the sense that the spatial period should be increased with the head, we conclude that $\lambda(y) = ctey^{B/A}$, and considering the results of [10], which indicates that the behavior is linear, we conclude that $B/A \approx 1$. Therefore, reiterating, the result shows that the force is proportional to the combination of the power velocity by another power of the hydraulic head y. And we write (15) as (16):

$$k\frac{(U)^{a-1+1/\alpha}}{(y)^{a-1}} \tag{16}$$

We restart the succession (4) which produces a representation of the force, which limit should produce the friction force between the particle and the fluid.

There are two important points: one corresponds to Stokes, the other to Ossen. For Stokes, the value $a = 3/2$ is required, because in (16) solves $a-1+1/2=1$, and for the other, $a = 5/2$, which solves $a-1+1/2=2$.

Furthermore, and in light of the present results we want to see its compatibility with those presented in [12], where it was found that the force on the particle is an expression of the type: $C\nu_a\phi'(d/2r)\frac{(U)^b}{(d)^{c_1}} = \Delta g(d)^{c_2}$, or $C\nu_a\phi'(d/2r)(U)^b = \Delta g(d)^c$ where ϕ is a function of the particle shape, which must be increasing and convex with respect to the aspect ratio $d/2r$, and $(d, r, U, \nu_\alpha, \Delta, g, C)$ are: particle size, distance from the particle, outer velocity, kinematic-viscosity, dimensionless variance of the specific weights of the material

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 3, May 2015

regarding the fluid and C is a set of units, respectively. The following relationships for the exponents of the derivative order and the power of the size of the particle are obtained: $a = b + 1 - 1/\alpha, \quad c_1 = c - b + 1/\alpha$. Then, for the Stokes' formula $b = 1, \quad c = 2$, which produces $a = 2 - 1/\alpha, \quad c_1 = 1 + 1/\alpha$, under the constraint $(1/2) < \alpha$.

For the Allens, $b = 3/2, \quad c = 3/2$, then $a = 5/2 - 1/\alpha, \quad c_1 = 1/\alpha$, with the restriction $(2/5) < \alpha \le 2$.

For Owens, $b = 2, \quad c = 1$, so $a = 3 - 1/\alpha, \quad c_1 = -1 + 1/\alpha$, and the restriction $(1/3) < \alpha < 1$. For Maza, $b = 2, \quad c = 7/10$, then $a = 3 - 1/\alpha, \quad c_1 = -(13/10) + 1/\alpha$, and the constraint $(1/3) < \alpha \le 10/13$.

Therefore, the exponent of the particle size increases and the order of the derivative decreases with the growth of the turbulence, which makes the smallest knot vortex. And we can say that the formulas of Stokes and Allen served from the turbulent sub-layer to the viscous, whereas those of Owens and Maza to turbulent regime.

In future enhancement, include the determination of the links between fractional Navier-Stokes equations and fluid interaction with objects at different scales.

VI. CONCLUSIONS

We have represented possible solutions of fractional Navier-Stokes equation as the limit of a sequence constructed as a linear combination of the Biot-Savart operator that convolves vorticity.

Also on the basis of the fractional Navier-Stokes equation, in the approach of the boundary layer, and postulating the description of coherent structures such as knots of vortices, we found the strength to measure the interaction between fluid and particle, as the limit of a succession of forces. Resultant force depends on, among others, by the regime of movement of fluid through the spatial occupancy ratio, as well, of course, by the external velocity and location of the particle flow.

When studying the consistency of these results with those shown in reference [1], we found a way to quantify the regime participation in the fluid motion in the various formulas for sedimentation velocity of solid particles. Taking as common features, when the turbulence is more intense the greater the power of the diameter of the Archimedes buoyancy, whereas when approaching the laminar regime, the lesser the same power. In addition, the more intense the turbulence the smaller vortex size knot and vice versa.

REFERENCES

[1] J. R. Mercado E., P. Guido A., J. Sánchez S., M. Íñiguez C. and A. González C. "Analysis of the Blasius' formula and the Navier-Stokes fractional equation," J. Klapp et al. (eds.), Fluid Dynamics in Physics, Engineering, and Environmental Applications, Environmental Science and Engineering, DOI: 10.1007/987-3-642-27723-8_44, 475-480 (Springer-Verlag Berlin Heidelberg, 2013).

[2] J. R. Mercado, P. Guido, W. Ojeda, J. Sánchez-Sesma and E. Olvera, "Saint-Venant fractional equation and hydraulic gradient," Journal of Math. and System Science **2** (8), pp. 494-503, 2012.

[3] A. Sommerfeld, "Mechanics of Deformable Bodies," Lectures on Theoretical Physics, Vol. **II**, Academic Press, New York, pp. 396, 1950.

[4] L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Pergamon Press, Oxford, 1987).

[5] A. J. Chorin, Vorticity and Turbulence (Springer, New York, 1994).

[6] A. Stasiak, V. Katritch and L.H. Kauffman (Editors), "Ideal Knots, Series on Knots and Everything," **Vol. 19**, World Scientific, Singapore et al.; Cap. 19, L.H. Kauffman; Cap. 12, L.H. Moffatt; Cap. 14, R. L. Ricca. (1998).

[7] S. Kida, "A vortex filament moving without change of form," J. Fluid Mech. **112**, 397–409 (1981).

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 4, Issue 3, May 2015

[8] J. R. Mercado, "Ecuación Blasius fraccional," Congreso Latinoamericano de Hidráulica, Punta del Este, Uruguay. Article published in the Proceedings under the reference 237, 2010.

[9] D. Kaftory, "Structures in the turbulent boundary layer and their interaction with particles," PhD Thesis. University of California, Santa Barbara, 1993.

[10] T. L. Doligalski, C. R. Smith and D. A. Walker, "Vortex interactions with walls," Annu. Rev. Fluid Mech. **26,** 573-616, 1994.

[11] J. R. Mercado, A. Aldama y F. Brambila, "Un problema inverso en las ecuaciones de Saint-Venant," Aportaciones Matemáticas, Serie Comunicaciones **32**, pp. 3-30, 2003.

[12] J. R. Mercado E., W. Ojeda B., P. Guido A. y G. Zetina D., "Velocidad de sedimentación e interacción fluido-partícula", 2do. Congreso Internacional de Investigación de Ciencias Básicas y Agronómicas, Universidad Autónoma de Chapingo, 21-22 nov. 2013.

AUTHOR BIOGRAPHY

J.R. Mercado-Escalante: Hydraulic Specialist in the Mexican Institute of Water Technology – IMTA, Mexico, affiliate to the Drainage and Irrigation Coordination. Dr. of Sciences (Mathematics) by the National Autonomous University of Mexico-UNAM, Master Degree in Mathematics by the Autonomous University of Puebla-BUAP, graduates in Physics by the National University of Colombia. Research areas: inverse problems, fractals and fractional derivatives, mathematical aspects of hydraulics.

P.A. Guido-Aldana: Hydraulic Specialist in the Mexican Institute of Water Technology – IMTA, Mexico, affiliate to the Professional Development Coordination. Dr. in Eng. and Master in Hydraulics by the National Autonomous University of Mexico-UNAM, Civil Eng. Associate Professor in the Faculty of Engineering of the UNAM. Research interest: river hydraulics, water planning and the implementation of PIV and LDA technologies in order to study open channel flows.

G. Zetina Domínguez: Specialist in the Federal Electricity Commission – CFE, Mexico. Master in Hydraulics by the National Autonomous University of Mexico-UNAM, Civil Eng. Research interest: river hydraulics, instrumentation, physical and numerical modeling.