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Pathway Transform Associated with H-Function and General Class of Polynomials

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Abstract—The object of this paper is to obtain P-transforms or Pathway transform of Fox’s H-function and a general class of polynomials. P-transforms are useful in reaction rate theory in astrophysics. Pathway transform is generalization of many integral transforms, H-function and general class of polynomials are general in nature. These results provide a number of new and known results on specializing of the parameters. Some particular cases are mentioned also.

Key words—P-transform, Fox’s H-function, General class of polynomials, Hermite polynomial, Laguerre polynomial.

I. INTRODUCTION

The P-transform or pathway transform is defined and represented in Kumar and Kilbas[11] as

$$(P_{\theta}^{\rho, \alpha, \beta} f)(x) = \int_0^{\infty} D_{\rho, \alpha}^{\theta, \beta}(xt) f(t) dt, x > 0 \tag{1.1}$$

where $D_{\rho, \alpha}^{\theta, \beta}(x)$ represents the kernel-function

$$D_{\rho, \alpha}^{\theta, \beta}(x) = \int_0^{\left[\frac{1}{1-\beta}\right]^{\frac{1}{\rho}}} y^{\theta-1} [1-a(1-\beta)y^{\rho}]^{\frac{1}{1-\theta}} e^{-xy^{\rho}} dy, x > 0, \tag{1.2}$$

With $\theta \in \mathbb{C}$, $\alpha > 0$, $\rho > 0$, $a > 0$, $\beta < 1$. When $D_{\rho, \alpha}^{\theta, \beta}(x)$ is given by (1.2), P-transform is called type-1 P-transform. If we use

$$D_{\rho, \alpha}^{\theta, \beta}(x) = \int_0^{\infty} y^{\theta-1} [1+a(\beta-1)y^{\rho}]^{\frac{1}{\theta-1}} e^{-xy^{\rho}} dy, x > 0, \tag{1.3}$$

For $\theta \in \mathbb{C}$, $\alpha > 0$, $\rho > \mathbb{R}$, $a > 0$, $\beta > 1$ in (1.1) then we obtain a type-2 P-transform. The P-transform of both types are defined in the space $L_{\alpha, r}(0, \infty)$ consisting of the Lebesgue measurable complex valued functions f for which

$$\|f\|_{\theta, r} = \left\{ \int_0^{\infty} |t^{\theta} f(t)|^r \frac{dt}{t} \right\}^{\frac{1}{r}} < \infty, \tag{1.4}$$

For $1 \leq r < \infty$, $\theta \in \mathbb{R}$. The P-transform of both the types are obtained by using the pathway model of Mathai [5], Mathai and Haubold [6]. If $\alpha=1$, $a=1$, and $\beta \rightarrow 1$, we have

$$\lim_{\beta \rightarrow 1} D_{\rho, 1}^{\theta, \beta}(x) = Z_{\rho}^{\theta}(x). \tag{1.5}$$

Where $Z_{\rho}^{\theta}(x)$ is the kernel function of the Krätzel transform, introduced by Krätzel[12] and given by

$$K_{\theta}^{(\rho)} f(x) = \int_0^{\infty} Z_{\rho}^{\theta}(xt) f(t) dt, x > 0, \tag{1.6}$$

$$\text{Where } Z_{\rho}^{\theta}(x) = \int_0^{\infty} y^{\theta-1} e^{-y^{\rho}-xy^{\rho-1}} dy. \tag{1.7}$$

The transform in (1.6) and its several modifications were considered by many authors. Glaeske et al. [14] considered a generalized version of the Krätzel transform and its compositions with fractional calculus operators



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on the spaces of $F_{p,u}$ and $F'_{p,u}$. Bonilla et al. [7,8] studied the Krätzel transform in the spaces $F_{p,u}$ and $F'_{p,u}$. Kilbas et al. [2] obtained the asymptotic representation for the modified Krätzel function. Kilbas et al. [3] studied the Krätzel function in (1.7) for all values of p and established it in terms of Fox's H-function. When $\alpha=1, a=1, \rho=1$ and $\beta \rightarrow 1$, P-transform of both types reduces to the Meijer transform. For $\alpha=1, a=1, \rho=1$ and $\beta \rightarrow 1$ along with x replaced by $\frac{t^2}{4}$ in (1.2) and (1.3), we get

$$\lim_{\rho \rightarrow 1} D_{1,1}^{\delta,\beta} \left(\frac{t^2}{4} \right) = 2 \left(\frac{t}{2} \right)^{\delta} K_{-\delta}(t), \quad (1.8)$$

Where $K_{-\delta}(t)$ is modified Bessel function of the third kind or the Mc-Donald function (see[4], sect. 7.2.2). Kilbas and Kumar [1] considered (1.3) for $\alpha=1$ and established its composition with fractional operators and represented it in terms of various generalized special functions.

The H- function introduced by Fox [10] will be represented and defined as follows

$$H_{p,q}^{m,n} \left[x \left| \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \theta(\xi) x^{\xi} d\xi, \quad (1.9)$$

where

$$\theta(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + A_j \xi)}{\prod_{j=1}^q \Gamma(1 - b_j + B_j \xi) \prod_{j=1}^p \Gamma(a_j - A_j \xi)}, \quad (1.10)$$

$i = \sqrt{-1}$, an empty product is interpreted as unity; $0 \leq m \leq q$; $0 \leq n \leq p$; $A_j (j=1, \dots, p)$ and $B_j (j=1, \dots, q)$ are positive numbers. L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j - B_j \xi) (j=1, \dots, m)$ lie to the right of it and those of $\Gamma(1 - a_j + A_j \xi) (j=1, \dots, n)$ lie to the left to it. Asymptotic expansions and analytic continuations of the H-function have been discussed by Braaksma[9]. A detailed account of the H-function is available in Srivastava et al. [16].

The general class of polynomials defined by Srivastava [15] in the following way

$$S_v^u [x] = \sum_{k=0}^{[v/u]} \frac{(-v)_{uk}}{k!} A_{v,k} x^k, \quad (1.11)$$

Where u, v are arbitrary positive integers and the coefficients $A_{v,k} (v, k > 0)$ are arbitrary constants, real or complex.

II. MAIN RESULTS

Theorem1. If $\in L_{\delta, \tau}(0, \infty)$, $z_1, z_2, \theta \in \mathbb{C}$, $\alpha > 0, b > 0, h > 0, \beta < 1$ be such that $\rho > 0$ in the case of a type-1 P-transform then

$$P_{\delta}^{\rho, \alpha, \beta} \left[H_{p,q}^{m,n} (z_1 t^b) S_v^u [z_2 t^h] \right] = \sum_{k=0}^{[v/u]} \frac{(-v)_{uk}}{k!} A_{v,k} z_2^k \frac{1}{\rho x^{h k + 1}} \frac{1}{[\alpha(1-\beta)]^{\frac{1}{\rho}}} \Gamma \left(\frac{1}{1-\beta} + 1 \right)$$



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$$H_{p+2, q+1}^{m, n+2} \left[\frac{z_1 x^b}{[a(1-\beta)]^{\frac{ab}{\rho}}} \middle| \begin{matrix} (a_j, A_j)_{1, n}, -hk, b, \left(1 - \frac{\theta + \alpha + \alpha h k}{\rho}, \frac{\alpha b}{\rho}\right), (a_j, A_j)_{n+1, p} \\ (b_j, B_j)_{1, q}, \left(\frac{1}{\beta-1} - \frac{\theta + \alpha + \alpha h k}{\rho}, \frac{\alpha b}{\rho}\right) \end{matrix} \right], \quad (2.1)$$

Where $\Re\left(\frac{1}{1-\beta} + 1\right) > 0$ and the coefficients $A_{v, k} (v, k > 0)$ are arbitrary constants, real or complex.

Theorem 2. If $f \in L_{\theta, r}(0, \infty)$, $z_1, z_2, \theta \in \mathbb{C}$, $\alpha > 0, b > 0, h > 0, \beta > 1$ be such that $\rho \in \mathbb{R}, \rho \neq 0$ in the case of a type-2 P-transform then

$$\begin{aligned} & P_{\theta}^{\alpha, \beta} \left[H_{p, q}^{m, n}(z_1 t^b) S_v^u [z_2 t^h] \right] \\ &= \\ & \sum_{k=0}^{[v/u]} \frac{(-v)_{u k}}{k!} A_{v, k} z_2^k \frac{1}{\rho x^{hk+1}} \frac{1}{[a(\beta-1)]^{\frac{ab}{\rho}}} \frac{1}{\Gamma\left(\frac{1}{\beta-1}\right)} H_{p+2, q+1}^{m+1, n+2} \left[\frac{z_1 x^b}{[a(\beta-1)]^{\frac{ab}{\rho}}} \middle| \begin{matrix} (a_j, A_j)_{1, n}, (-hk, b), \left(1 - \frac{\theta + \alpha + \alpha h k}{\rho}, \frac{\alpha b}{\rho}\right), (a_j, A_j)_{n+1, p} \\ (b_j, B_j)_{1, m}, \left(\frac{1}{\beta-1} - \frac{\theta + \alpha + \alpha h k}{\rho}, \frac{\alpha b}{\rho}\right), (b_j, B_j)_{m+1, q} \end{matrix} \right] \end{aligned} \quad (2.2)$$

Where $\Re\left(\frac{1}{\beta-1}\right) > 0$ and the coefficients $A_{v, k} (v, k > 0)$ are arbitrary constants, real or complex.

Proof. To obtain (2.1) we consider type-1 P-transform using (1.1) express Fox's H-function and general class of polynomial with the help of (1.9), (1.11), we have

$$\begin{aligned} & P_{\theta}^{\alpha, \beta} \left[H_{p, q}^{m, n}(z_1 t^b) S_v^u [z_2 t^h] \right] = \int_0^{\infty} D_{\rho, \alpha}^{\theta, \beta}(xt) H_{p, q}^{m, n}(z_1 t^b) S_v^u [z_2 t^h] dt \\ &= \frac{1}{2\pi i} \int_0^{\infty} \int_0^{\left[\frac{1}{a(1-\beta)}\right]^{\frac{1}{\rho}}} y^{\theta-1} [1-a(1-\beta)y^{\rho}]^{\frac{1}{1-\beta}} e^{-xt} y^{-\alpha} \\ & \int_L \theta(\xi) z_1^{\xi} t^{b\xi+hk} d\xi \sum_{k=0}^{[v/u]} \frac{(-v)_{u k}}{k!} A_{v, k} z_2^k dy dt. \end{aligned}$$

Changing the order of integrations and series, we get

$$\begin{aligned} &= \sum_{k=0}^{[v/u]} \frac{(-v)_{u k}}{k!} A_{v, k} z_2^k \\ & \frac{1}{2\pi i} \int_L \theta(\xi) z_1^{\xi} \int_0^{\left[\frac{1}{a(1-\beta)}\right]^{\frac{1}{\rho}}} y^{\theta-1} [1-a(1-\beta)y^{\rho}]^{\frac{1}{1-\beta}} \int_0^{\infty} e^{-xy} y^{-\alpha} t^{b\xi+hk} dt dy d\xi. \end{aligned}$$

Taking $xy^{-\alpha} = w$ and using gamma function, we obtain

$$\begin{aligned} &= \sum_{k=0}^{[v/u]} \frac{(-v)_{u k}}{k!} A_{v, k} z_2^k \\ & \frac{1}{2\pi i} \int_L \theta(\xi) \frac{z_1^{\xi}}{x^{b\xi+hk+1}} \Gamma(b\xi+hk+1) \int_0^{\left[\frac{1}{a(1-\beta)}\right]^{\frac{1}{\rho}}} y^{\theta+\alpha b\xi+\alpha h k+\alpha-1} [1-a(1-\beta)y^{\rho}]^{\frac{1}{1-\beta}} dy d\xi. \end{aligned}$$

Now, solving inner integral with the help of beta function and then reinterpreting the result Mellin-Barnes contour integral in terms of Fox's H-function, we get the desired result.



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On the similar lines, the proof of result (2.2) can be established using the definition of H-function(1.9) and type-2 P-transform(1.1).

III. SPECIAL CASES

On taking $a = 1, \alpha = 1$ and $\beta \rightarrow 1$ in result (2.1), we have

$$(a) \lim_{\beta \rightarrow 1} P_{\theta}^{\alpha, 1, \beta} \left[H_{p, q}^{m, n}(z_1 t^b) S_{\nu}^u [z_2 t^h] \right] = \sum_{k=0}^{[v/u]} \frac{(-v)_{uk}}{k!} A_{\nu, k} z_2^k$$

$$\frac{1}{\rho x^{hk+1}} H_{p+2, q}^{m, n+2} \left[\frac{z_1}{x^b} \left| \begin{matrix} (a_j, A_j)_{1, n'} \\ (b_j, B_j)_{1, q} \end{matrix} \right. \begin{matrix} (-hk, b) \\ \left(1 - \frac{\theta+hk+1}{\rho}, \frac{b}{\rho}\right) \end{matrix}, (a_j, A_j)_{n+1, p} \right]. \quad (3.1)$$

By applying $u = 2$ in (2.1), (2.2) and (3.1), general class of polynomial reduces to Hermite polynomials [13] by setting

$$S_{\nu}^u [x] = x^{v/2} H_{\nu} \left(\frac{1}{2\sqrt{x}} \right), \text{ in this case } A_{\nu, k} = (-1)^k, \text{ we get}$$

$$(b) P_{\theta}^{\alpha, \alpha, \beta} \left[H_{p, q}^{m, n}(z_1 t^b) (z_2 t^h)^{v/2} H_{\nu} \left(\frac{1}{2\sqrt{z_2 t^h}} \right) \right] = \sum_{k=0}^{[v/2]} \frac{(-v)_{2k}}{k!} (-z_2)^k \frac{1}{\rho x^{hk+1}} \frac{1}{[a(1-\beta)]^{\frac{\theta+\alpha+\alpha hk}{\rho}}}$$

$$\Gamma \left(\frac{1}{1-\beta} + 1 \right) H_{p+2, q+1}^{m, n+2} \left[\frac{z_1 x^b}{[a(1-\beta)]^{\frac{\alpha b}{\rho}}} \left| \begin{matrix} (a_j, A_j)_{1, n'} \\ (b_j, B_j)_{1, q'} \end{matrix} \right. \begin{matrix} (-hk, b) \\ \left(1 - \frac{\theta+\alpha+\alpha hk}{\rho}, \frac{\alpha b}{\rho}\right) \end{matrix}, (a_j, A_j)_{n+1, p} \right]. \quad (3.2)$$

$$(c) P_{\theta}^{\alpha, \alpha, \beta} \left[H_{p, q}^{m, n}(z_1 t^b) (z_2 t^h)^{v/2} H_{\nu} \left(\frac{1}{2\sqrt{z_2 t^h}} \right) \right] = \sum_{k=0}^{[v/2]} \frac{(-v)_{2k}}{k!} (-z_2)^k \frac{1}{\rho x^{hk+1}} \frac{1}{[a(\beta-1)]^{\frac{\theta+\alpha+\alpha hk}{\rho}}} \frac{1}{\Gamma \left(\frac{1}{\beta-1} \right)}$$

$$H_{p+2, q+1}^{m+1, n+2} \left[\frac{z_1 x^b}{[a(\beta-1)]^{\frac{\alpha b}{\rho}}} \left| \begin{matrix} (a_j, A_j)_{1, n'} \\ (b_j, B_j)_{1, m} \end{matrix} \right. \begin{matrix} (-hk, b) \\ \left(1 - \frac{\theta+\alpha+\alpha hk}{\rho}, \frac{\alpha b}{\rho}\right) \end{matrix}, (a_j, A_j)_{n+1, p} \right]. \quad (3.3)$$

$$(d) \lim_{\beta \rightarrow 1} P_{\theta}^{\alpha, 1, \beta} \left[H_{p, q}^{m, n}(z_1 t^b) (z_2 t^h)^{v/2} H_{\nu} \left(\frac{1}{2\sqrt{z_2 t^h}} \right) \right] = \sum_{k=0}^{[v/2]} \frac{(-v)_{2k}}{k!} (-z_2)^k$$

$$\frac{1}{\rho x^{hk+1}} H_{p+2, q}^{m, n+2} \left[\frac{z_1}{x^b} \left| \begin{matrix} (a_j, A_j)_{1, n'} \\ (b_j, B_j)_{1, q} \end{matrix} \right. \begin{matrix} (-hk, b) \\ \left(1 - \frac{\theta+hk+1}{\rho}, \frac{b}{\rho}\right) \end{matrix}, (a_j, A_j)_{n+1, p} \right]. \quad (3.4)$$

Results in (b), (c) and (d) are valid under the same conditions as are required for (2.1), (2.2) and (3.1) respectively. Putting $u = 1$ in (2.1), (2.2) and (3.1) general class of polynomial reduces to Laguerre polynomials [13] by setting

$$S_{\nu}^u [x] = L_{\nu}^s(x), \text{ in this case } A_{\nu, k} = \binom{v+s}{s+1}_k \frac{1}{(s+1)_k}, \text{ we get}$$



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$$(e) \mathcal{P}_{\theta}^{\rho, \alpha, \beta} [H_{p, q}^{m, n}(z_1 t^b) L_v^s(z_2 t^h)] = \sum_{k=0}^{[v]} \binom{v+s}{v-k} \frac{(-z_2)^k}{k!} \frac{1}{\rho x^{hk+1}} \frac{1}{[a(1-\beta)]^{\frac{\theta+\alpha+\alpha hk}{\rho}}} \Gamma\left(\frac{1}{1-\beta} + 1\right) H_{p+2, q+1}^{m, n+2} \left[\frac{z_1 x^{-b}}{[a(1-\beta)]^{\frac{\alpha b}{\rho}}} \middle| \begin{matrix} (a_j, A_j)_{1, n}, (-h k, b), \left(1 - \frac{\theta+\alpha+\alpha hk}{\rho}, \frac{\alpha b}{\rho}\right), (a_j, A_j)_{n+1, p} \\ (b_j, B_j)_{1, q}, \left(\frac{1}{\beta-1} - \frac{\theta+\alpha+\alpha hk}{\rho}, \frac{\alpha b}{\rho}\right) \end{matrix} \right]. \quad (3.5)$$

$$(f) \mathcal{P}_{\theta}^{\rho, \alpha, \beta} [H_{p, q}^{m, n}(z_1 t^b) L_v^s(z_2 t^h)] = \sum_{k=0}^{[v]} \binom{v+s}{v-k} \frac{(-z_2)^k}{k!} \frac{1}{\rho x^{hk+1}} \frac{1}{[a(\beta-1)]^{\frac{\theta+\alpha+\alpha hk}{\rho}}} \Gamma\left(\frac{1}{\beta-1}\right) H_{p+2, q+1}^{m+1, n+2} \left[\frac{z_1 x^{-b}}{[a(\beta-1)]^{\frac{\alpha b}{\rho}}} \middle| \begin{matrix} (a_j, A_j)_{1, n}, (-h k, b), \left(1 - \frac{\theta+\alpha+\alpha hk}{\rho}, \frac{\alpha b}{\rho}\right), (a_j, A_j)_{n+1, p} \\ (b_j, B_j)_{1, m}, \left(\frac{1}{\beta-1} - \frac{\theta+\alpha+\alpha hk}{\rho}, \frac{\alpha b}{\rho}\right), (b_j, B_j)_{m+1, q} \end{matrix} \right]. \quad (3.6)$$

$$(g) \lim_{\beta \rightarrow 1} \mathcal{P}_{\theta}^{\rho, 1, \beta} [H_{p, q}^{m, n}(z_1 t^b) L_v^s(z_2 t^h)] = \sum_{k=0}^{[v]} \binom{v+s}{v-k} \frac{(-z_2)^k}{k!} \frac{1}{\rho x^{hk+1}} H_{p+2, q}^{m, n+2} \left[\frac{z_1}{x^b} \middle| \begin{matrix} (a_j, A_j)_{1, n}, (-h k, b), \left(1 - \frac{\theta+h k+1}{\rho}, \frac{b}{\rho}\right), (a_j, A_j)_{n+1, p} \\ (b_j, B_j)_{1, q} \end{matrix} \right]. \quad (3.7)$$

Results in (e), (f) and (g) are valid under the same conditions as are required for (2.1), (2.2) and (3.1) respectively. The results derived in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions.

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