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Unsteady Free Convection Flow of a Couple Stress Fluid - A State Space Approach

G. Shantha[1] ,M.Sunitha[2] , B.Shanker[3]
[1]Department of Mathematics and Humanities, Mahatma Gandhi Institute of Technology, Hyderabad-500078, India
[2]Department of Mathematics, University College for Women, Osmania University, Hyderabad-500 001, India
[3]Department of Mathematics, Osmania University, Hyderabad- 506 004, India

Abstract: ***The equations of motion describing the unsteady, laminar, incompressible free convection flow of a couple stress fluid which is bounded by a vertical infinite plate are considered. Using Laplace transform technique, the equations are cast into matrix form and the flow field variables are obtained through state space approach in Laplace transform domain. The obtained expressions are inverted making use of a numerical approach and the variation of the relevant flow field variables (viz.,) velocity and temperature distributions are presented through graphs for different values of the physical and material parameters. The problem is solved for two specific cases: (i) a thermal shock is given to the vertical plate; (ii) a plane distribution of continuous heat sources is located on the plate.***

Keywords: **Couple stress fluid, State space approach, Laplace transform, Numerical inversion.**

I. INTRODUCTION

The couple stress fluid theory proposed by Stokes (1984) is a simple generalization of the classical Navier-Stokes theory [6]. This theory provides for the polar effects such as sustenance of couple stresses and body couples in a fluid medium. The rotation associated with each particle in the fluid is the actual vorticity vector at any point in the medium. This is the simplest theory that shows all the important features and effects of the couple stresses in a fluid medium and results in equations that are similar to the Navier-Stokes equations. Second order gradient of the velocity vector is introduced into the stress constitutive equations. The stress tensor is no longer symmetric and the antisymmetric part of the stress tensor and the trace of couple stress tensor are determined by the constitutive equations. In this paper, we consider an unsteady free convection flow of an incompressible couple stress fluid which is bounded by an infinite vertical plane. The temperature conditions are allowed to be general with initial zero conditions. The problem is solved making use of Laplace Transforms in time. The resulting equations are cast into a matrix form with the use of an unknown state vector consisting of the velocity, temperature and their gradients in the Laplace Transform domain. The resulting matrix equation is solved through a state space approach [5]. This approach enables us to use the methodology of modern control theory. This method is successfully used by Helmy (2000)[2], Helmy et al (2002)[3] and Devakar and Iyengar (2008)[1].

II. STATEMENT OF THE PROBLEM

Consider an unsteady, laminar, incompressible flow of a couple stress fluid which is bounded by an infinite vertical plate. Let all the properties of the fluid be assumed constant except that the influence of the density variation with temperature is considered only in the body force term.

A point O on the plate is taken as origin; a vertical line through O (a point on the plate) is taken as x-axis and the perpendicular through O to the plate is taken as y-axis. The equations governing the unsteady free convection flow with the assumptions that the velocity vector $\bar{q} = \left(u(y,t),\, v(y,t),0\right)$ and temperature T are functions of distance y and time t are given by

$$\frac{\partial v}{\partial y} = 0 \qquad (1)$$

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$$\rho\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\eta_1\frac{\partial^4 u}{\partial y^4}+\rho g\beta(T-T_w) \quad (2)$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}+\tau_0\left(\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}\right)\right)=\frac{1}{\rho c_p}\left(\lambda\frac{\partial^2 T}{\partial y^2}+Q+\tau_0\frac{\partial Q}{\partial t}\right) \quad (3)$$

where

μ is the viscosity coefficient ,

η_1 is the couple stress viscosity coefficient ,

$\frac{\lambda}{\rho c_p}$ is the thermal diffusivity coefficient ,

β is the coefficient of volumetric expansion ,

Q is intensity of heat source and

τ_0 is the thermal relaxation time.

In the momentum equation (2) above, the density variation is taken into account by the Boussinesq approximation. In the energy equation (3), the viscous and Joules dissipation are neglected since in a free convection flow they are, in general, assumed to be very small. In this equation, the relaxation time effect is considered as a result of the generalized Fourier's law application. The continuity equation (1) implies that the velocity component v is a function of time or a constant. We shall assume here that v = 0.

Let us introduce the following non-dimensional scheme:

$$y = Ly', \ u=\frac{\upsilon u'}{L}, \ \theta=\frac{T-T_\infty}{T_w-T_\infty}, \ t=\frac{L^2 t'}{\upsilon} \ \text{and} \ Q=\frac{(T_w-T_\infty)\lambda Q'}{L^2} \quad (4)$$

The kinematic viscosity υ , Prandtl number P and the Grashoff number G are defined through

$$\upsilon=\frac{\mu}{\rho}, \ P=\frac{C_p\rho\upsilon}{\lambda} \ \text{and} \ G=\frac{g\beta L^3}{\upsilon^2}(T_w-T_\infty) \quad (5)$$

Since the velocity component v= 0, in view of the above nondimensionalization scheme, dropping the primes, the non dimensional equations governing the velocity and temperature fields are given by

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial y^2}-\frac{1}{\lambda_1^2}\frac{\partial^4 u}{\partial y^4}+G\theta \quad (6)$$

$$P\left(\frac{\partial\theta}{\partial t}+\tau_0\frac{\partial^2\theta}{\partial t^2}\right)=\frac{\partial^2\theta}{\partial t^2}+\left((Q+\tau_0\frac{\partial Q}{\partial t})\right) \quad (7)$$

Where

$$\lambda_1^2=\frac{\mu L^2}{\eta_1} \quad (8)$$

These equations are to be solved for u and θ subject to the relevant boundary conditions. For the present problem, these are

$$u(0,t)=0 \quad (9)$$

$$\frac{\partial u(0,t)}{\partial y}=0 \quad (10)$$

and

$$\theta(0,t)=Q(t) \quad (11)$$

Taking the Laplace transforms of the above equations (6) and (7) , we get

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Volume 3, Issue 1, January 2014

$$\frac{d^4\bar{u}}{dy^4} - \lambda_1^2 \frac{d^2\bar{u}}{dy^2} + s\,\lambda_1^2\,\bar{u} - \lambda_1^2\,G\bar{\theta} = 0 \qquad (12)$$

and

$$\frac{d^2\bar{\theta}}{dy^2} = ns\bar{\theta} - \frac{n}{P}\,\bar{Q} \qquad (13)$$

where, $\bar{\theta}(y,s)$, $\bar{u}(y,s)$ and $\bar{q}$ are respectively the Laplace transform of $\theta(y,t)$, $u(y,t)$ and Q with

$$n = P(1+\tau_0 s) \qquad (14)$$

Taking the Laplace transforms of the boundary conditions when the nature of temperature distribution on the vertical plate is known, we solve the equations (12) and (13) with these boundary conditions and obtain the expressions for the velocity and temperature distributions in the Laplace transform domain. By inverting these to the original time domain, we can obtain the velocity and temperature distributions.

III. SOLUTION THROUGH STATE SPACE APPROACH

We shall rewrite the equations (6) and (7) in matrix notation adopting the state space approach (Ogata 1967)[5].

Let $u,\ \frac{\partial u}{\partial y} = u_1,\ \frac{\partial u_1}{\partial y} = u_2,\ \frac{\partial u_2}{\partial y} = u_3,\quad \theta,\ \frac{\partial \theta}{\partial y} = \theta'$ (15)

be taken as physical variables in the (y , t) domain and $\bar{u}$, $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, $\bar{\theta}$, $\bar{\theta}'$ be the corresponding Laplace transforms of the physical variables in (y, s) domain .
The equations (12) and (13) together now take the matrix form

$$\frac{d}{dy}\bar{V}(y,s) = \bar{A}(s)\bar{V}(y,s) + \bar{B}(y,s) \qquad (16)$$

where

$$\bar{V}(y,s) = \begin{bmatrix}\bar{u} & \bar{\theta} & \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{\theta}'\end{bmatrix}^T \qquad (17)$$

$$\bar{B}(y,s) = -\frac{n}{P}\bar{Q}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}^T \qquad (18)$$

and the matrix

$$\bar{A}(s) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\lambda_1^2 s & \lambda_1^2 G & 0 & \lambda_1^2 & 0 & 0 \\ 0 & ns & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (19)$$

The equation (16) has its formal solution in the form

$$\bar{V}(y,s) = \exp(\bar{A}(s)y)\,\{\,\bar{V}(0,s) + \int_0^y \exp(-A(s)z\,B(z,s)\,dz \qquad (20)$$

This formal expression of $\bar{V}(y,s)$ can be explicitly written down if we can calculate the matrix exp $[\bar{A}(s)y]$.
The techniques employed in state space analysis allow us to handle the evaluation of exp [A(s)y] as below :

The characteristic equation of the matrix $\bar{A}(s)$ is given below

$$k^6 - ak^4 + bk^2 - c = 0 \qquad (22)$$

Where

ISSN: 2319-5967
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Volume 3, Issue 1, January 2014

$$a = \lambda_1^2 + ns$$
$$b = \lambda_1^2 s\,(1+n) \qquad (23)$$
$$c = \lambda_1^2 n s^2$$

The equation (22) is same as

$$(k^2 - ns)\left(k^4 - \lambda_1^2 k^2 + \lambda^2 s\right) = 0 \qquad (24)$$

which can be written as

$$(k^2 - k_1^2)\,(k^2 - k_2^2)(k^2 - k_3^2) = 0 \qquad (25)$$

Where

$$k_1^2 = ns,\; k_2^2 + k_3^2 = \lambda_1^2,\, k_2^2 k_3^2 = \lambda_1^2\, s \qquad (26)$$

and roots of (25) are pair wise distinct. Using Cayley –Hamilton theorem, we have

$$A^6 - a\,A^4 + bA^2 - cI = 0 \qquad (27)$$

and hence A^6 and higher powers of A can be written in terms of $I, A, A^2, A^3, A^4 and\ A^5$ where I is the unit matrix of order 6.

By definition, the matrix $\exp\left[\overline{A}(s)y\right]$ is given by

$$\exp\left[\overline{A}(s)y\right] = L^*(y,s) = \sum_{i=0}^{\infty} a_i^* \frac{(A(s)y)^i}{i!} \qquad (28)$$

and when we use (27) , this can be written in the form

$$\exp\left[\overline{A}(s)y\right] = L^*(y,s) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3 + a_4 A^4 + a_5 A^5 \qquad (29)$$

The expression for the matrix $L^*(y,s)$ will be completely determined only when the scalars $a_0, a_1, a_2, a_3, a_4, a_5$ are determined. However we know that if k is any characteristic root of $\overline{A}(s)$, then

$$\exp[ky] = a_0 + a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5$$

and hence we have the following six equations

$$\begin{aligned}
\exp[k_1 y] &= a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3 + a_4 k_1^4 + a_5 k_1^5 \\
\exp[-k_1 y] &= a_0 - a_1 k_1 + a_2 k_1^2 - a_3 k_1^3 + a_4 k_1^4 - a_5 k_1^5 \\
\exp[k_2 y] &= a_0 + a_1 k_2 + a_2 k_2^2 + a_3 k_2^3 + a_4 k_2^4 + a_5 k_2^5 \\
\exp[-k_2 y] &= a_0 - a_1 k_2 + a_2 k_2^2 - a_3 k_2^3 + a_4 k_2^4 - a_5 k_2^5 \\
\exp[k_3 y] &= a_0 + a_1 k_3 + a_2 k_3^2 + a_3 k_3^3 + a_4 k_3^4 + a_5 k_3^5 \\
\exp[-k_3 y] &= a_0 - a_1 k_3 + a_2 k_3^2 - a_3 k_3^3 + a_4 k_3^4 - a_5 k_3^5
\end{aligned} \qquad (30)$$

The solution of the above system of linear equations is given by

$$\begin{aligned}
a_0 &= -H\left[k_2^2 k_3^2 C_1 + k_3^2 k_1^2 C_2 + k_1^2 k_2^2 C_3\right] \\
a_1 &= -H\left[k_2^2 k_3^2 B_1 + k_3^2 k_1^2 B_2 + k_1^2 k_2^2 B_3\right] \\
a_2 &= H\left[(k_2^2 + k_3^2)C_1 + (k_3^2 + k_1^2)C_2 + (k_1^2 + k_2^2)C_3\right] \\
a_3 &= H\left[(k_2^2 + k_3^2)B_1 + (k_3^2 + k_1^2)B_2 + (k_1^2 + k_2^2)B_3\right] \\
a_4 &= -H\left[C_1 + C_2 + C_3\right], \quad a_5 = H\left[B_1 + B_2 + B_3\right]
\end{aligned} \qquad (31)$$

where

$$H = \frac{1}{(k_2^2 - k_3^2)(k_3^2 - k_1^2)\,(k_1^2 - k_2^2)}$$
$$C_1 = (k_2^2 - k_3^2)\cosh(k_1 y)$$
$$C_2 = (k_3^2 - k_1^2)\cosh(k_2 y) \qquad (32)$$

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International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 3, Issue 1, January 2014

$$C_3 = (k_1^2 - k_2^2)\cosh(k_3 y),\ B_1 = \left(\frac{k_2^2 - k_3^2}{k_1}\right)\sinh(k_1 y)$$

$$B_2 = \left(\frac{k_3^2 - k_1^2}{k_2}\right)\sinh(k_2 y),\ B_3 = \left(\frac{k_1^2 - k_2^2}{k_3}\right)\sinh(k_3 y)$$

By straight forward calculation of A, A^2, A^3, A^4, A^5 and using these along with equations (31) and (29), we get the matrix $L^*(y,s)$ completely. The problem is concerned with the half space $y \geq 0$. The expression of $L^*(y,s)$ involves terms of the form sinh(*ky*) and cosh(*ky*). If k has positive real part, these will be unbounded at infinity. We want the solution to be finite at infinity. Hence we shall replace each sinh(*ky*) with (-1/2)exp(-*ky*) and cosh (*ky*) with (1/2)exp(-*ky*) . Let us denote the matrix $L^*(y,s)$ after this replacement by *L*(*y*,*s*) .The elements L_{ij} of this matrix *L* are seen to be

$$L_{11} = \frac{1}{2(k_2^2 - k_3^2)}\left[-k_3^2 e^{-k_2 y} + k_2^2 e^{-k_3 y}\right]$$

$$L_{12} = -\frac{GH(k_2^2 + k_3^2)}{2}\left[(k_2^2 - k_3^2)e^{-k_1 y} + (k_3^2 - k_1^2)e^{-k_2 y} + (k_1^2 - k_2^2)e^{-k_3 y}\right]$$

$$L_{13} = \frac{1}{2k_2 k_3(k_2^2 - k_3^2)}\left[k_3^3 e^{-k_2 y} - k_2^3 e^{-k_3 y}\right]$$

$$L_{14} = \frac{1}{2(k_2^2 - k_3^2)}\left[e^{-k_2 y} - e^{-k_3 y}\right]$$

$$L_{15} = \frac{1}{2k_2 k_3(k_2^2 - k_3^2)}\left[k_2 e^{-k_3 y} - k_3 e^{-k_2 y}\right]$$

$$L_{16} = \frac{GH(k_2^2 + k_3^2)}{2k_1 k_2 k_3}\left[k_2 k_3(k_2^2 - k_3^2)e^{-k_1 y} + k_1 k_3(k_3^2 - k_1^2)e^{-k_2 y} + k_1 k_2(k_1^2 - k_2^2)e^{-k_3 y}\right]$$

$$L_{21} = L_{23} = L_{24} = L_{25} = 0\ ;\ L_{22} = \frac{1}{2}e^{-k_1 y};\ \ L_{26} = -\frac{1}{2k_1}e^{-k_1 y}$$

$$L_{31} = \frac{-k_2 k_3}{2(k_2^2 - k_3^2)}\left[k_2 e^{-k_3 y} - k_3 e^{-k_2 y}\right]$$

$$L_{32} = \frac{GH(k_2^2 + k_3^2)}{2}\left[k_1(k_2^2 - k_3^2)e^{-k_1 y} + k_2(k_3^2 - k_1^2)e^{-k_2 y} + k_3(k_1^2 - k_2^2)e^{-k_3 y}\right]$$

$$L_{33} = L_{11};\ L_{34} = \frac{1}{2(k_2^2 - k_3^2)}\left[k_3 e^{-k_2 y} - k_2 e^{-k_3 y}\right]$$

$$L_{35} = L_{14};\ L_{36} = L_{12};\ L_{41} = \frac{-k_2^2 k_3^2}{2(k_2^2 - k_3^2)}\left[e^{-k_2 y} - e^{-k_3 y}\right]$$

$$L_{42} = \frac{-GH(k_2^2 + k_3^2)}{2}\left[k_1^2(k_2^2 - k_3^2)e^{-k_1 y} + k_2^2(k_3^2 - k_1^2)e^{-k_2 y} + k_3^2(k_1^2 - k_2^2)e^{-k_3 y}\right]$$

$$L_{43} = \frac{k_2 k_3}{2(k_2^2 - k_3^2)}\left[k_3 e^{-k_2 y} - k_2 e^{-k_3 y}\right];\ \ L_{44} = L_{11} + \lambda_1^2 L_{14};\ \ L_{45} = L_{13} + \lambda_1^2 L_{15}$$

$$L_{46} = L_{32}\ ;\ L_{51} = -\lambda_1^2 s L_{34};\ L_{52} = \lambda_1^2 G L_{34} + ns L_{32}\ ;\ L_{53} = -\lambda_1^2 s L_{14}$$

$$L_{54} = \lambda_1^2 L_{34} - \lambda_1^2 s L_{15}\ ;\ L_{55} = L_{44}\ ;\ L_{56} = L_{42}\ ;\ L_{61} = L_{63} = L_{64} = L_{65} = 0$$

$$L_{62} = ns L_{26}\ ;\ L_{66} = L_{22} \qquad (33)$$

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Volume 3, Issue 1, January 2014

Now the solution $\bar{V}(y,s)$ which is finite at $y=\infty$ is given by (20) where exp(A(s)y) is to be replaced by L . Thus $\bar{V}(y,s)$ is completely determined in Laplace Transform domain. We present hereunder two specific applications.

IV. AN APPLICATION TO THERMAL SHOCK PROBLEM

Let us assume that in the general problem considered above, we specifically give a thermal shock of the form
$\theta(0,t)=\theta_0 H(t)$
to the vertical plate y = 0 , where θ_0 is a constant and H(t) is Heaviside unit step function. (i.e) suddenly at time t=0, the temperature of the plate y = 0 is made θ_0 (a constant).
This amounts to the assumption that there are no heat sources acting inside the fluid. Hence, from equation (20), it follows that $\bar{V}(y,s)$ is given by

$$\begin{aligned}\bar{V}(y,s) &= \exp[\bar{A}(s)y]\bar{V}(0,s) \\ &= L(y,s)\bar{V}(0,s)\end{aligned} \quad (34)$$

Let $u(y,0)$, $\frac{\partial u}{\partial y}(y,0)$ be assumed to be zero. Let us adopt the state space approach described above to obtain the velocity and temperature distributions.
When we take Laplace transform for the initial conditions, we get

$$\bar{u}(0,s)=0,\ \frac{\partial \bar{u}}{\partial y}(0,s)=0, \bar{\theta}(0,s)=\frac{\theta_0}{s} \quad (35)$$

The components $\bar{u}_2(0,s)$, $\bar{u}_3(0,s)$ *and* $\bar{\theta}'(0,s)$ can be obtained by using the above three conditions in (34). These equations give

$$\bar{u}_2(0,s) = \frac{G(k_2^2+k_3^2)}{(k_1+k_2)(k_3+k_1)}\frac{\theta_0}{s} \quad (36)$$

$$\bar{u}_2(0,s) = \frac{-G(k_2^2+k_3^2)(k_1+k_2+k_3)}{(k_1+k_2)(k_3+k_1)}\frac{\theta_0}{s} \quad (37)$$

$$\bar{\theta}'(0,s) = -k_1\frac{\theta_0}{s} \quad (38)$$

Now using equation (17) and equations (34) - (37), we get

$$\bar{u}(y,s) = \frac{-G(k_2^2+k_3^2)(k_2+k_3)}{(k_1^2-k_2^2)(k_2^2-k_3^2)(k_3^2-k_1^2)}\{(k_2-k_3)\exp(-k_1y)+(k_3-k_1)\exp(-k_2y)+(k_1-k_2)\exp(-k_3y)\}\frac{\theta_0}{s} \quad (39)$$

$$\bar{\theta}(y,s) = \exp(-k_1y)\frac{\theta_0}{s} \quad (40)$$

Thus the velocity and temperature distributions are explicitly determined in the Laplace transform domain. To get the velocity and temperature distributions in the physical domain , we have to take the inverse Laplace transform of $\bar{u}(y,s)$ *and* $\bar{\theta}(y,s)$. An explicit determination of these inverse transforms is difficult in view of the nature of k_1, k_2, k_3 which are themselves functions of s and other parameters . It is possible to invert numerically using a convenient numerical inversion technique.

V. NUMERICAL RESULTS AND DISCUSSION

The numerical inversion of $\bar{u}(y,s)$ and $\bar{\theta}(y,s)$ is carried out making use of the numerical inversion method proposed by Honig and Hirdes (1984)[4]. Computations are performed for different values of the parameters $\theta_0=1, \tau_0=0.0,0.1,0.2$ and $\lambda_1=0.1,0.2,0.3$.

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Volume 3, Issue 1, January 2014

Figure (1) represents the temperature distribution. It is noticed that the temperature has its maximum value on the wall y = 0 and decreases with increase in value of y. Figure (2) shows the variation of velocity for different values of τ_0 for Grashoff number G = 2. As the relaxation time parameter increases velocity decreases. Further for any fixed τ_0, the velocity gradually increases, attains a maximum and decreases latter. For Grashoff number G = -2 the variation of velocity is displayed in Figure (3). Here as τ_0 increases velocity increases. Further for fixed τ_0 as we move away from the vertical plate velocity decreases attains a minimum and then gradually increases to zero. Figures (4) and (5) display the variation of velocity with distance with Grashoff number 2 and -2 respectively for couple stress parameter λ_1 = 0.1 ,0.2 ,0.3 . Here again for G = 2 as λ_1 increases, couple stress viscosity η decreases, as can be expected, the velocity increases. Further as we move away from the fixed plate, for any fixed λ_1 the velocity gradually increases, attains a maximum and gradually decreases. For Grashoff number G = -2 as λ_1 increases, velocity decreases. Also for any fixed λ_1 velocity decreases attain s a minimum and increases to zero. As observed by Helmy (2000) for the case of micro polar fluid, in the present case of couple stress fluid also, we observe that the effect of cooling and heating by free convection currents are occurring when G>0 and G<0 respectively. The behavior of the velocity here is also qualitatively similar to the one observed by Helmy (2000)[2] for the micro polar fluid.

VI. AN APPLICATION TO PLANE DISTRIBUTION OF HEAT SOURCE

In this section, we assume that there is a plane distribution of continuous heat sources located at the plate *y* = 0. Let the intensity of the heat source at any point *y* at time *t* be taken as

$$Q(y,t) = Q_0\, H(t)\,\delta(y) \tag{41}$$

Where *H* (*t*) is Heaviside's function $\delta(y)$ is Dirac delta function
Taking Laplace transform of this, we get

$$\bar{Q}(y,s) = \frac{Q_0}{s}\delta(y) \tag{42}$$

Using this in equation (18) , from equation (20) we get

$$\bar{V}(y,s) = L(y,s)\,[\,\bar{V}(0,s) + \bar{F}(s)] \tag{43}$$

where

$$\bar{F}(s) = -\frac{nQ}{Ps}\begin{bmatrix} -L_{16}^0 & -\frac{1}{2k_1} & 0 & -L_{46}^0 & 0 & \frac{1}{2} \end{bmatrix} \tag{44}$$

using the properties of integrals involving Dirac Delta function . The expressions for L_{16}^0 *and* L_{46}^0 can be obtained from (28) and these are given by

$$L_{16}^0 = \frac{-G(k_2^2+k_3^2)(k_1+k_2+k_3)}{2k_1k_2k_3(k_1+k_2)(k_2+k_3)(k_3+k_1)} \tag{45}$$

$$L_{46}^0 = \frac{G(k_2^2+k_3^2)}{2(k_1+k_2)(k_2+k_3)(k_3+k_1)} \tag{46}$$

The equation (37) gives the solution $\bar{V}(y,s)$ in Laplace Transform domain for $y \ge 0$ in terms of the vector $\bar{F}(s)$ which has arisen due to the applied heat source . The components of the vector $\bar{V}(y,s)$ can be calculated using the boundary conditions

$$u(0,t)=0 \;\text{ and }\; \frac{\partial u(0,t)}{\partial y}=0 \tag{47}$$

which yield

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Volume 3, Issue 1, January 2014

$$\bar{u}(0,s) = 0 \tag{48}$$

$$\bar{u}'(0,s) = 0 \tag{49}$$

in Laplace transform domain . As in Helmy el al (2002)[3], consider a cylinder of unit base whose axis is perpendicular to the plane source of heat and whose bases lie on opposite sides of it. Taking the limit as the height of the cylinder tends to zero, with the assumption that there is no heat flux through the lateral surface, we get

$$q(0,t) = \frac{Q_0}{2} H(t) \tag{50}$$

Using the generalized Fourier's law of heat conduction in the non-dimensional form, we obtain

$$q + \tau_0 \frac{\partial q}{\partial t} = -\left(\frac{\partial \theta}{\partial y}\right)_{y=0} \tag{51}$$

Taking Laplace transform of (50) we obtain

$$\bar{\theta}'(0,s) = -(1+\tau_0 s)\frac{Q_0}{2s} = -\frac{nQ_0}{2ps} \tag{52}$$

The equations (43), (44) and (47) give three components of the vector $\bar{V}(0,s)$. The other three components can be obtained by putting y =0 on either side of the equation (43). Using (48), (49) and (52) in

$$\bar{V}(0,s) = L(0,s)\,[\,\bar{V}(0,s) + \bar{F}(s)\,] \tag{53}$$

we get

$$\bar{u}_3(0,s) = \frac{-G(k_2^2+k_3^2)(k_1+k_2+k_3)}{k_1(k_1+k_2)(k_3+k_1)}\frac{nQ_0}{2Ps} \tag{54}$$

$$\bar{\theta}(0,s) = \frac{nQ_0}{2Psk_1} \tag{55}$$

and

$$L_{32}^0\,\bar{\theta}(0,s) - \frac{1}{2(k_2+k_3)}\bar{u}_2(0,s) - \frac{nQ_0}{Ps}\frac{G(k_2^2+k_3^2)(k_1+k_2+k_3)}{2k_1(k_2+k_3)^2(k_1+k_2)(k_3+k_1)} = 0 \quad (56)$$

Thus using (53) and (54) in (55) we get

$$\bar{u}_2(0,s) = \frac{-G(k_2^2+k_3^2)(2k_1+k_2+k_3)}{2k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)}\frac{nQ_0}{Ps} \tag{56}$$

The remaining three equations that arise from equation (52) are consistent with the above three equations. Thus the components $\bar{u}_2(0,s)$,$\bar{u}_3(0,s)$ $and\ \bar{\theta}(0,s)$ are completely determined.

Now using equation (42), we get

$$\bar{u}(y,s) = -G(k_2^2+k_3^2)\left\{\begin{array}{l} \dfrac{1}{k_1(k_1^2-k_2^2)(k_3^2-k_1^2)}e^{-k_1 y} \\ -\dfrac{1}{k_1k_2(k_2^2-k_3^2)(k_1+k_2)}\left[\dfrac{k_3(k_1+k_2+k_3)}{(k_2+k_3)(k_3+k_1)}+\dfrac{k_1}{(k_2-k_1)}\right]e^{-k_2 y} \\ +\dfrac{1}{k_1k_3(k_2^2-k_3^2)(k_3+k_1)}\left[\dfrac{k_2(k_1+k_2+k_3)}{(k_2+k_3)(k_2+k_1)}+\dfrac{k_1}{(k_3-k_1)}\right]e^{-k_3 y} \end{array}\right\}\frac{nQ_0}{2Ps}$$

(57)

$$\bar{\theta}(y,s) = \frac{nQ_0}{2Psk_1}e^{-k_1 y} \tag{58}$$

Thus the above two equations give the velocity and temperature distributions in the case of plane distribution of a heat source.

VII. NUMERICAL RESULTS AND DISCUSSION

As in the earlier case, here again we invert $\bar{u}(y,s)$ and $\bar{\theta}(y,s)$ numerically using the procedure of Honig and Hirdes (1984)[4]. Figure (6) indicates the variation of temperature distribution with respect to distance y for different values of the relaxation time parameter τ_0. As τ_0 increases the temperature increases for any fixed y in general. Further as we move away from the plate the temperature decreases. In figures (7) and (8) we present the variation of velocity for Grashoff number G = 2 , -2 with respect to distance for different values of τ_0. For G negative, for a fixed τ_0, as y increases , velocity decreases , attains a minimum and increases to 0 as y further increases. For G positive, for any τ_0, the velocity increases as y increases , attains a maximum and then tends to zero. For τ_0 = 0.0, 0.1, 0.2 there seems to be no significant variation in velocity for a fixed time t = 1. Variation of velocity with respect to distance for Grashoff number G= 2 , -2 for different values of λ_1 is observed in figures (9) and (10) . For τ_0 = 0.1, G=2, P=5, t=1 for any λ_1, the velocity increases as y increases, attains a maximum and then tends to zero . For any fixed y, as λ_1 increases (i.e. as couple stress viscosity decreases) the velocity increases. For G=-2, for any y, as λ_1 increases, velocity decreases, attains a minimum and then increases to zero.

VIII. CONCLUSIONS

The unsteady, laminar, incompressible, free convection flow of a couple stress fluid bounded by a vertical infinite plate is solved using Laplace transform technique without any restrictions on either the parameters or velocity and temperature distributions. The transformed equations are solved using state space approach, which is popular in solving problems in modern control theory in electrical engineering. This is solved exactly in Laplace transform domain and the transformed flow variables are inverted making use of a numerical inversion technique.

Two problems pertaining to (i) a thermal shock given to the plate at time t=0 and (ii) a plane distribution of heat source at the plate are solved. The variation of temperature and velocity with respect to couple stress parameter, relaxation time parameter and Grashoff number is shown through graphs.

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APPENDIX

FIGURE CAPTIONS

Figure (1): Temperature distribution for thermal shock problem

Figure (2): Velocity distribution for G = 2

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Figures

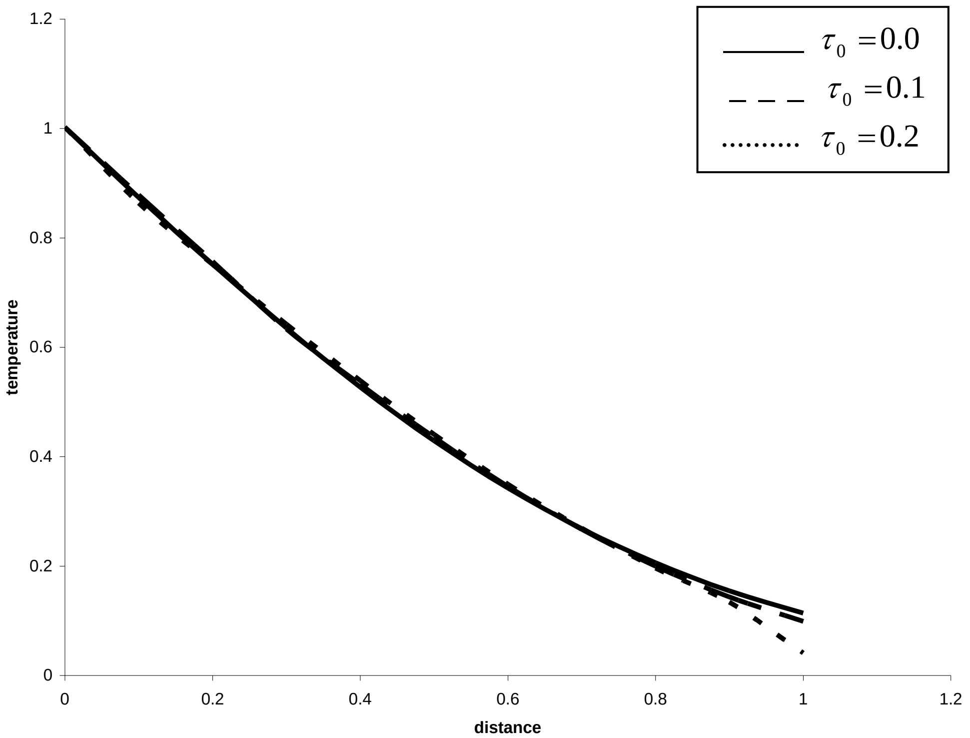

Fig (1): Temperature distribution for thermal shock problem

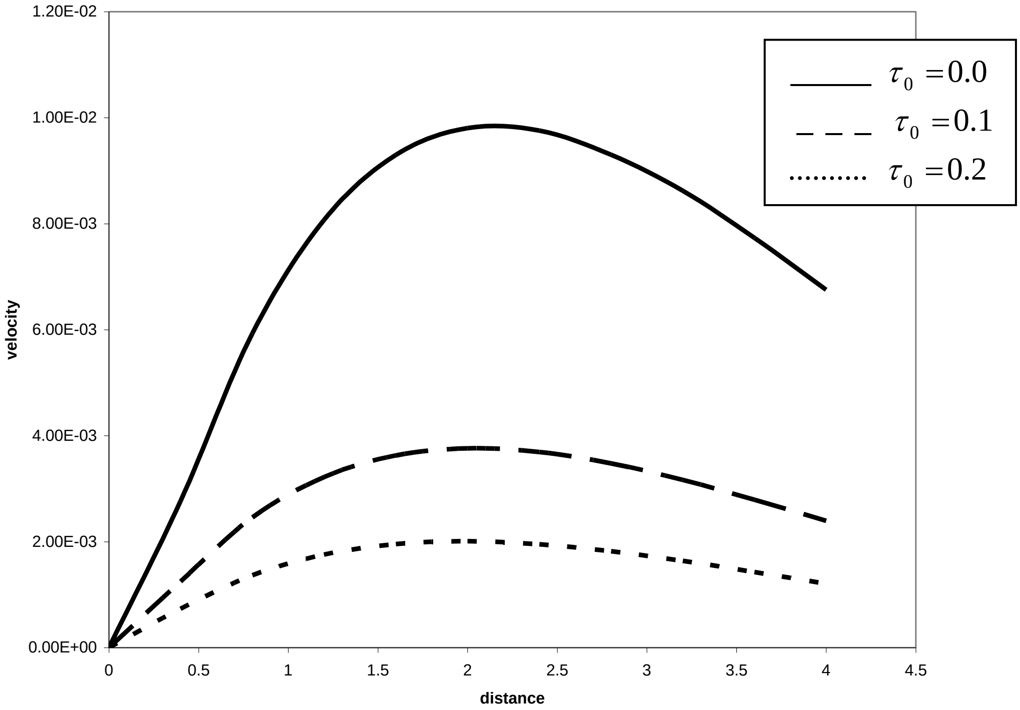

Fig (2): Velocity distribution for G = 2

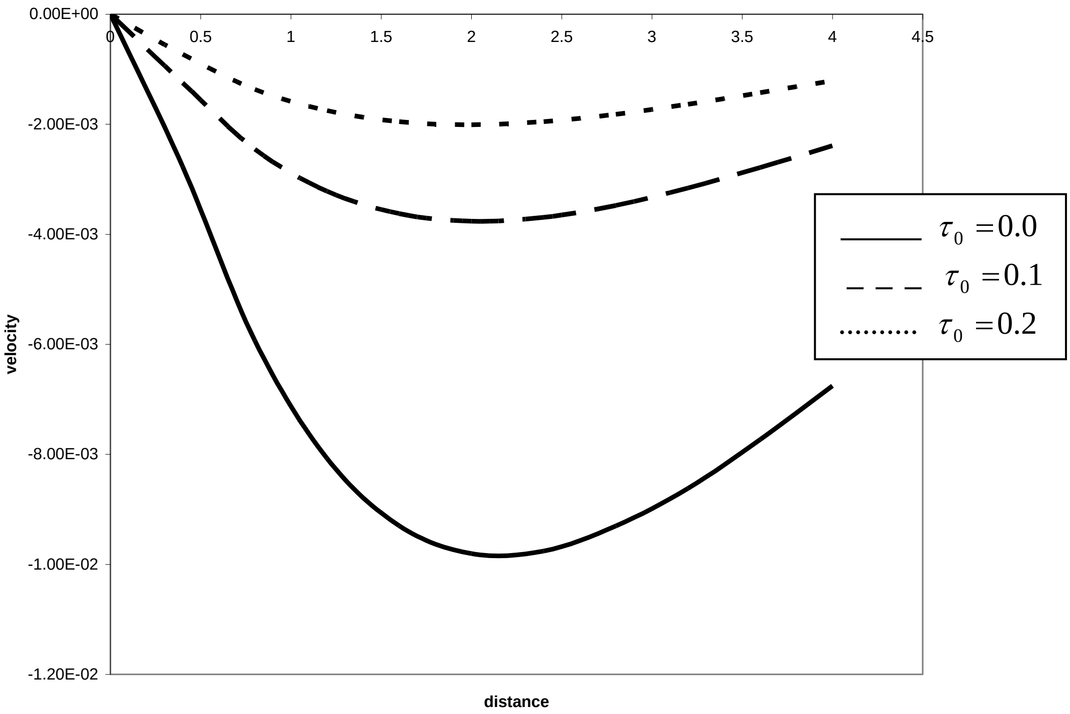

Fig (3): Velocity distribution for G = -2

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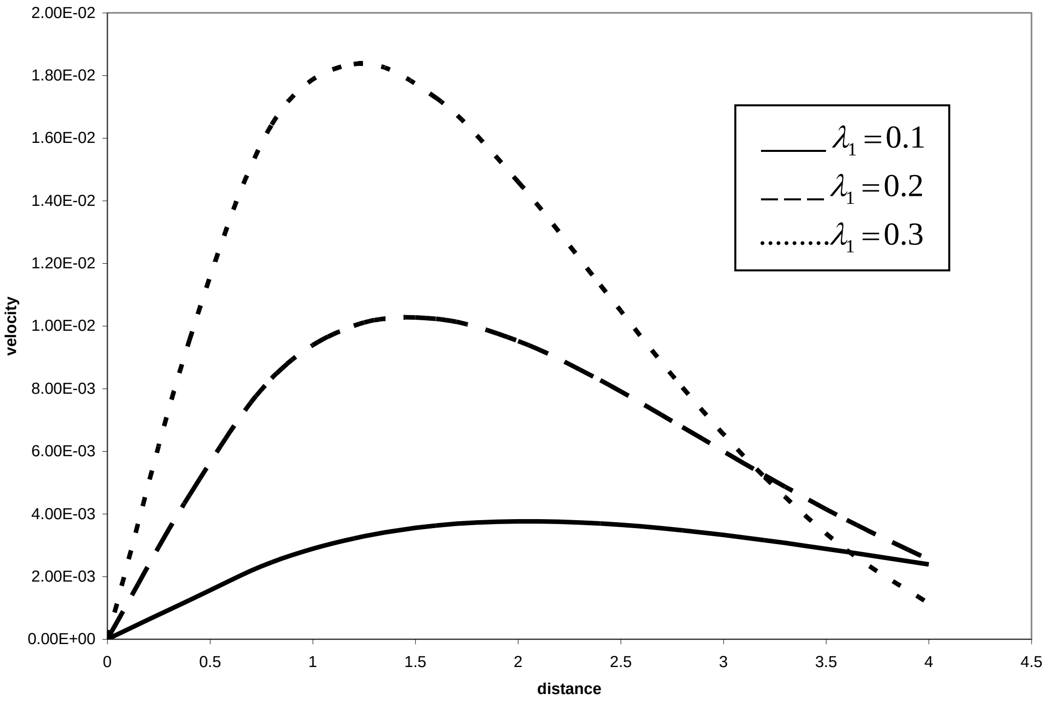

Fig (4): Velocity distribution for G = 2

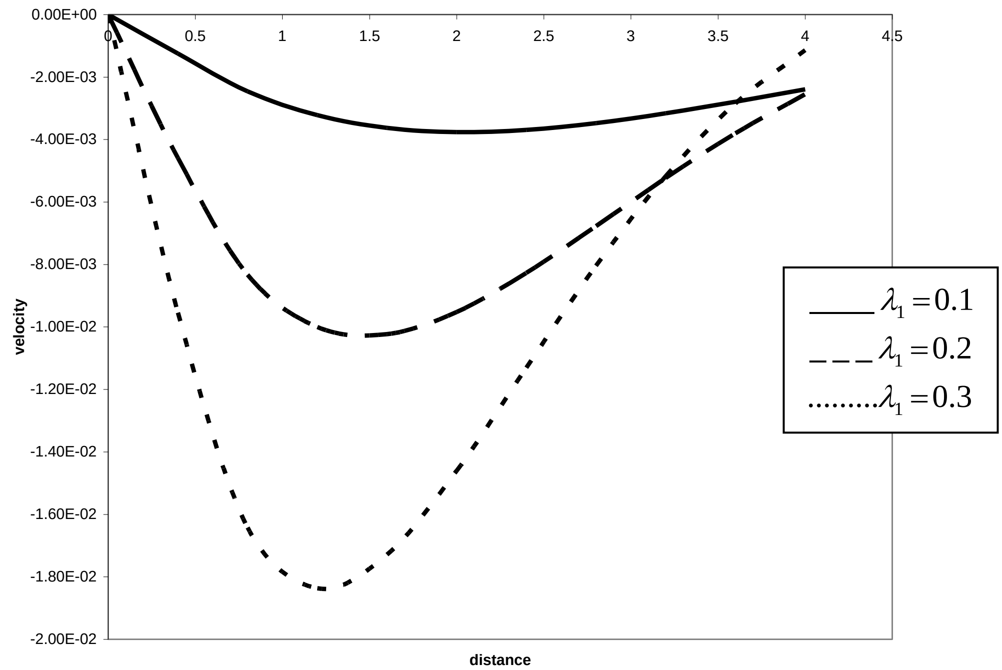

Fig (5): Velocity distribution for G = -2

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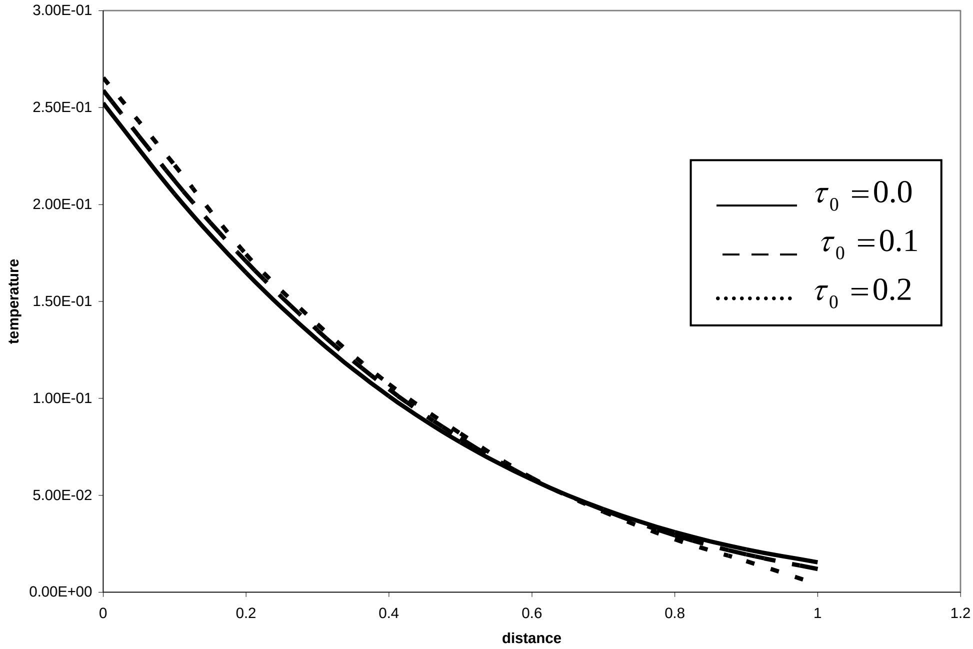

Fig (6): Temperature distribution for plane distributions of heat source

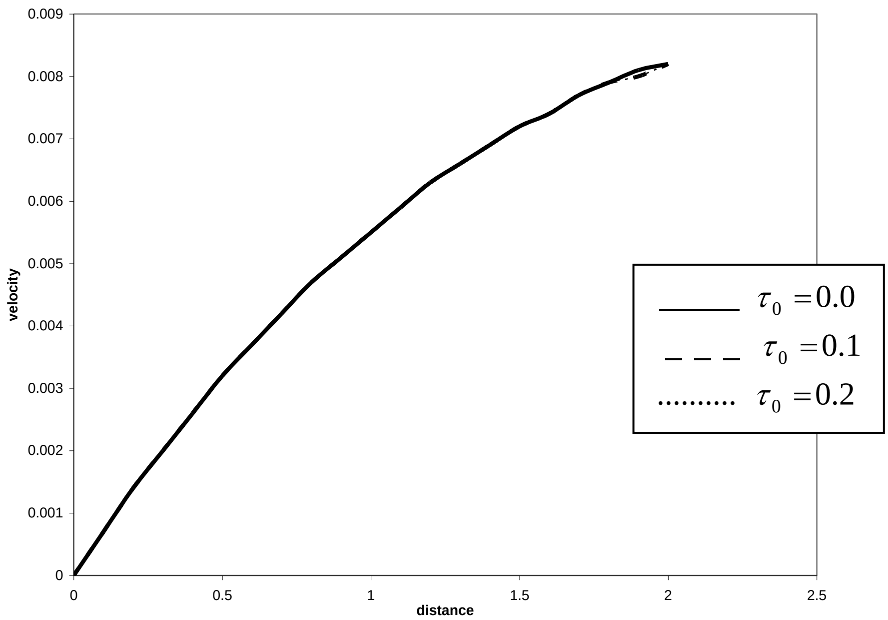

Fig (7): Velocity distribution for G = 2

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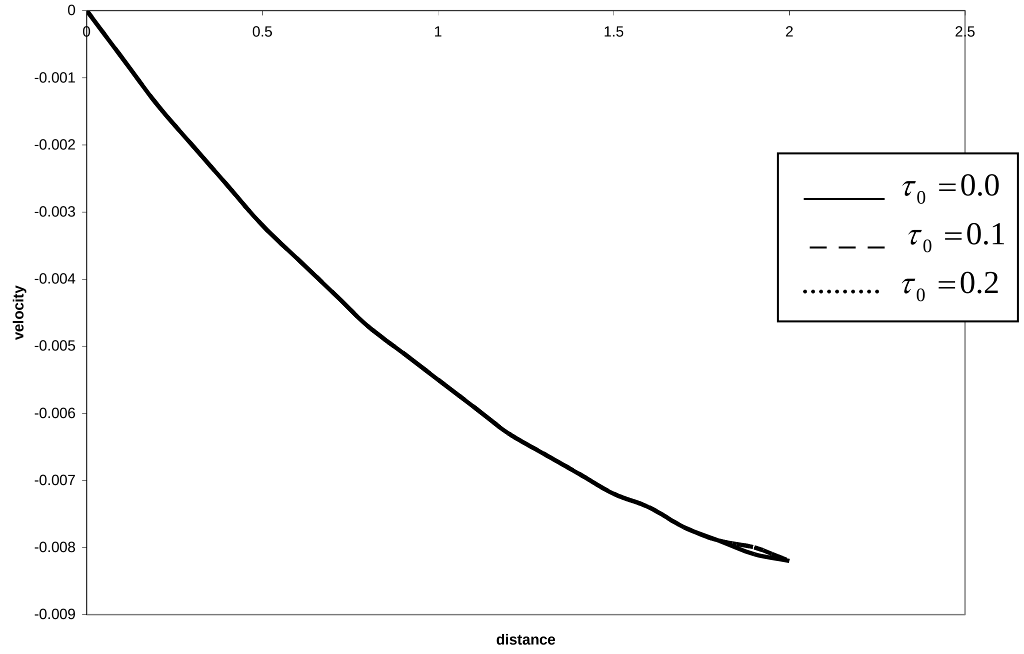

Fig (8): Velocity distribution for G = -2

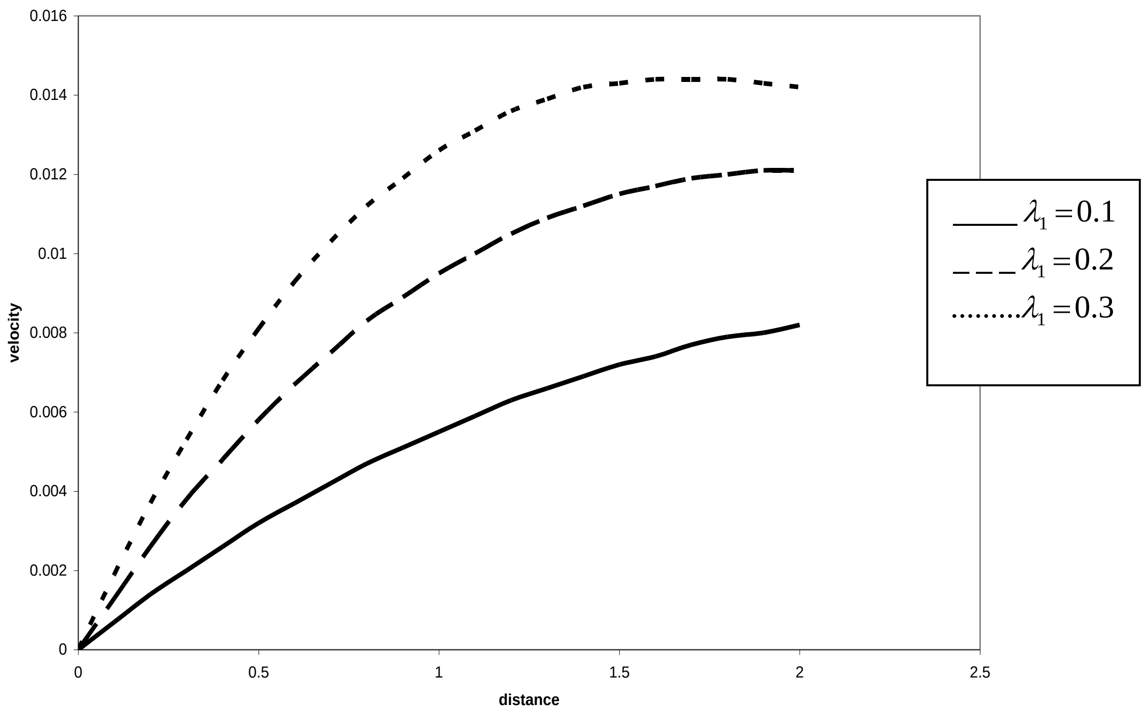

Fig (9): Velocity distribution for G = 2

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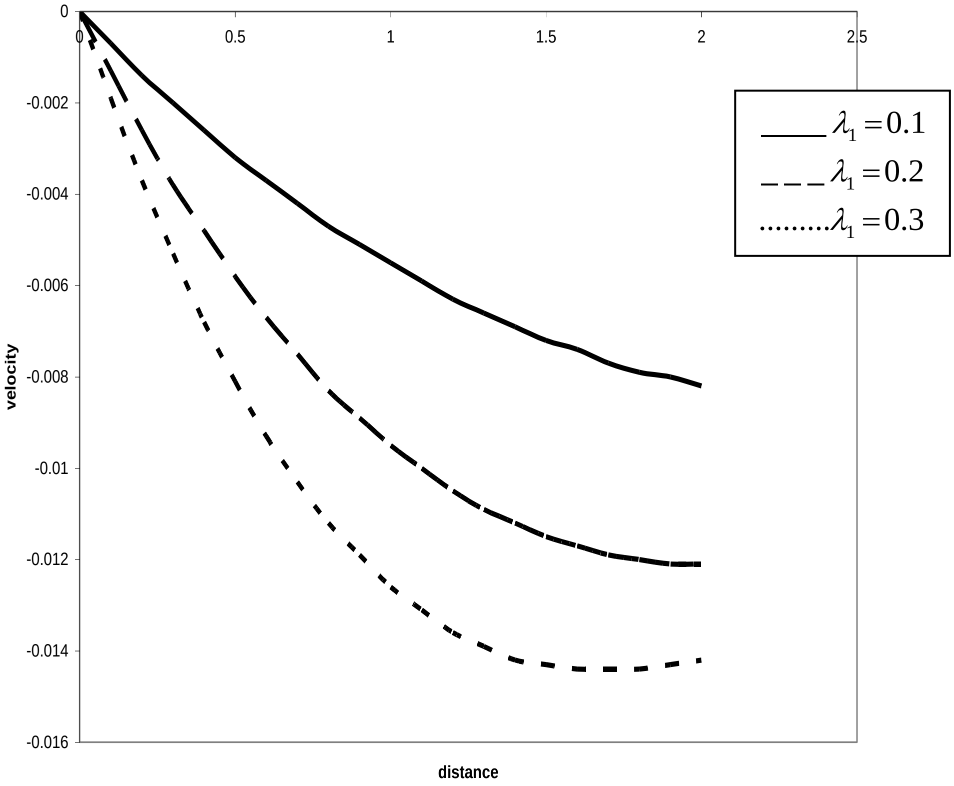

Fig (10): Velocity distribution for G = -2