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Stability Analysis of Mathematical Syn-Ecological Model Comprising of Prey-Predator, Host-Commensal, Mutualism and Neutral Pairs-IV (One of the Four Species are Washed out States)

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Abstract* - This investigation deals with a mathematical model of a four species (S_1, S_2, S_3 and S_4) Syn-Ecological system (one of the four species are washed out states). S_2 is a predator surviving on the prey S_1. The predator S_2 is a commensal to the host S_3. The pairs S_2 and S_4, S_1 and S_3 are neutral. The mathematical model equations characterizing the syn-ecosystem constitute a set of four first order non-linear coupled differential equations. There are in all sixteen equilibrium points. Criteria for the stability of four of the sixteen equilibrium points: One of the four species are washed out states only are established in this paper. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

I. INTRODUCTION

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiodology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmacokinetics are some of those disciplines. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of every one. Mathematical modeling of ecosystems was initiated by Lotka [9] and by Volterra [18]. The general concept of modeling has been presented in the treatises of Meyer [11], Cushing [4], Paul Colinvaux [11], Freedman [5], Kapur [6, 7]. The ecological interactions can be broadly classified as Prey-Predation, Competition, and Mutualism and so on. N.C. Srinivas [17] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [8] has investigated the two species prey-predator models. Stability analysis of competitive species was carried out by Archana Reddy [3] while Acharyulu [1, 2] investigated Ammensalism between two species. Recently local stability analysis for a two-species ecological mutualism model has been investigated by present author et al [12, 13, 14, 15, 16]. Example for S_1, S_2, S_3 and S_4 are Insects, Insectivorous Plants (nephantis, drosera etc.), VAM associated with the plant roots, Soil bacteria respectively.

II. BASIC EQUATIONS

The model equations for a four species multi-system are given by a set of four non-linear ordinary differential equations as

(i) For S_1**:** The Prey of S_1 and Neutral to S_3

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 \quad \text{.... (2.1)}$$

(ii) For S_2**:** The Predator surviving on S_1 and Commensal to S_3

$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_2N_1 + a_{23}N_2N_3 \quad \text{.... (2.2)}$$

(iii) For S_3**:** The Host of S_2 and Mutual to S_4

$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 + a_{34}N_3N_4 \quad \text{.... (2.3)}$$

(iv) For S_4**:** Mutual to S_3 and Neutral to S_2

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$$\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^{\,2} + a_{43} N_4 N_3 \qquad \text{.... (2.4)}$$

with the following notation.

N_i (t): Population strengths of the species S_i at time t, i=1, 2, 3, 4.
a_i : The natural growth rates of S_i, i = 1,2,3,4
a_{12},a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{13} : Coefficient for commensal for S_1 due to the Host S_3
$a_{34,}$ a_{43} : Mutually interaction between S_3 and S_4

K_i: $\frac{a_i}{a_{ii}}$: Carrying capacities of $S_{i,}$ i=1, 2, 3, 4.

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24} are assumed to be non-negative constants.

III. EQUILIBRIUM STATES

The system under investigation has sixteen equilibrium states defined by

$$\frac{dN_i}{dt} = 0,\ i = 1, 2, 3, 4 \qquad \text{...... (3.1)}$$

are given in the following table.

I. Fully washed out state:

E1: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$

II. States in which three of the four species are washed out and fourth is surviving

E2: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$

E3: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

E4: $\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$

E5: $\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$

III. States in which two of the four species are washed out while the other two are surviving

E6: $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33} a_{44} - a_{34} a_{43}}, \overline{N_4} = \frac{a_3 a_{43} + a_4 a_{33}}{a_{33} a_{44} - a_{34} a_{43}}$

This state exists only when $a_{33} a_{44} - a_{34} a_{43} > 0$

E7: $\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$

E8: $\overline{N_1} = 0, \overline{N_2} = \frac{a_3}{a_{22}} \frac{a_{23}}{a_{33}} + \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

E9: $\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$

E10: $\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

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E$_{11}$: $\overline{N_1} = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_2} = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_3} = 0, \overline{N_4} = 0$

This state exists only when $a_1 a_{22} - a_2 a_{12} > 0$

IV. States in which one of the four species is washed out while the other three are surviving

E$_{12}$: $\overline{N_1} = 0, \overline{N_2} = \frac{a_{23}(a_4 a_{34} + a_3 a_{44})}{a_{22}(a_{33}a_{44} - a_{34}a_{43})} + \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33}a_{44} - a_{34}a_{43}},$

$\overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33}a_{44} - a_{34}a_{43}}$

This state exists only when $a_{33}a_{44} - a_{34}a_{43} > 0$

E$_{13}$: $\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33}a_{44} - a_{34}a_{43}}$

This state exists only when $(a_{33}a_{44} - a_{34}a_{43}) > 0$

E$_{14}$: $\overline{N_1} = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_2} = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11}a_{22} + a_{12}a_{21}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$

This state exists only when $a_1 a_{22} - a_2 a_{12} > 0$

E$_{15}$: $\overline{N_1} = \frac{\beta_4}{\beta_1}, \overline{N_2} = \frac{\beta_5}{\beta_1}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

Where

$\beta_1 = a_{33}(a_{11}a_{22} + a_{12}a_{21}), \beta_4 = a_{33}(a_1 a_{22} - a_2 a_{12}) - a_3 a_{23} a_{12}$

$\beta_5 = a_{33}(a_1 a_{21} + a_2 a_{11}) + a_3 a_{23} a_{11}$

This state exists only when $\beta_4 > 0$

V. The co-existent state (or) Normal steady state

E$_{16}$: $\overline{N_1} = \frac{\gamma_1 + a_{12}a_{23}\gamma_2}{\gamma_3(a_{33}a_{44} - a_{34}a_{43})}, \overline{N_2} = \frac{\gamma_4 + a_{11}a_{23}\gamma_2}{\gamma_3(a_{33}a_{44} - a_{34}a_{43})},$

$\overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33}a_{44} - a_{34}a_{43}}$

Where

$\gamma_1 = (a_1 a_{22} + a_2 a_{12})(a_{33}a_{44} - a_{34}a_{43}), \gamma_2 = a_3 a_{44} + a_4 a_{34}$

$\gamma_3 = a_{11}a_{22} + a_{12}a_{21}, \gamma_4 = (a_1 a_{21} - a_2 a_{11})(a_{33}a_{44} - a_{34}a_{43})$

This state exists only when $(a_1 a_{21} - a_2 a_{11}) > 0$ *and* $(a_{33}a_{44} - a_{34}a_{43}) > 0$.

The present paper deals with one of the four species are washed out states only. The stability of the other equilibrium states will be presented in the forth coming communications.

IV. STABILITY OF ONE OF THE FOUR SPECIES WASHED OUT EQUILIBRIUM STATES:
V. (SL. NOS 12,13,14,15 IN THE ABOVE EQUILIBRIUM STATES)

A. *Stability of the Equilibrium State E$_{12}$:*

Let us consider small deviations $u_1(t), u_2(t), u_3(t), u_4(t)$ from the steady state i.e.

$$N_i(t) = \bar{N}_i + u_i(t),\ i = 1,2,3,4 \qquad \text{--- (4.1.1)}$$

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Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = w_1 u_1 \qquad \ldots (4.1.2)$$

$$\frac{du_2}{dt} = a_{21}\overline{N_2}u_1 - w_3 u_2 + a_{23}\overline{N_2}u_3 \qquad \ldots (4.1.3)$$

$$\frac{du_3}{dt} = -a_{33}\overline{N_3}u_3 + a_{34}\overline{N_3}u_4 \qquad \ldots (4.1.4) \qquad \frac{du_4}{dt} = a_{43}\overline{N_4}u_3 - a_{44}\overline{N_4}u_4 \qquad \ldots (4.1.5)$$

Here $w_1 = a_1 - a_{12}\overline{N_2}$, $w_3 = a_2 + a_{23}\overline{N_3}$ ... (4.1.6)

The characteristic equation of which is

$$(\lambda - w_1)(\lambda + w_3)\left[\lambda^2 + (a_{33}\overline{N_3} + a_{44}\overline{N_4})\lambda + (a_{33}a_{44} - a_{34}a_{43})\overline{N_3}\,\overline{N_4}\right] = 0 \qquad \ldots (4.1.7)$$

The characteristic roots of (4.1.6) are

$$\lambda = w_1, \lambda = -w_3, \lambda = \frac{-(a_{33}\bar{N}_3 + a_{44}\bar{N}_4) \pm \sqrt{(a_{33}\bar{N}_3 - a_{44}\bar{N}_4)^2 + 4a_{34}a_{43}\bar{N}_3\bar{N}_4}}{2}$$

Case (A): If $w_1 < 0$ [i.e. $a_1 < a_{12}\overline{N_2}$]

Here $w_1, -w_3$ are negative and the other two roots are also negative.

Hence the equilibrium state is **stable**.

The solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are

$$u_1 = u_{10}e^{w_1 t} \qquad \ldots (4.1.8)$$

$$u_2 = \left[u_{20} - \frac{a_{21}\overline{N_2}u_{10}}{(w_1 + w_3)} - \frac{a_{23}\overline{N_2}(P_1 + P_2)}{(\lambda_3 + w_3)}\right]e^{-w_3 t} + \frac{a_{21}\overline{N_2}u_{10}}{(w_1 + w_3)}e^{w_1 t} + \frac{a_{23}\overline{N_2}(P_1 e^{\lambda_3 t} + P_2 e^{\lambda_4 t})}{(\lambda_3 + w_3)} \qquad \ldots (4.1.9)$$

$$u_3 = \left[\frac{u_{30}\left(\lambda_3 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_3 - \lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{30}\left(\lambda_4 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_4 - \lambda_3}\right]e^{\lambda_4 t} \qquad \ldots (4.1.10)$$

$$u_4 = \left[\frac{u_{40}\left(\lambda_3 + a_{33}\bar{N}_3\right) + u_{30}a_{43}\bar{N}_4}{\lambda_3 - \lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{40}\left(\lambda_4 + a_{33}\bar{N}_3\right) + u_{30}a_{43}\bar{N}_4}{\lambda_4 - \lambda_3}\right]e^{\lambda_4 t} \qquad \ldots (4.1.11)$$

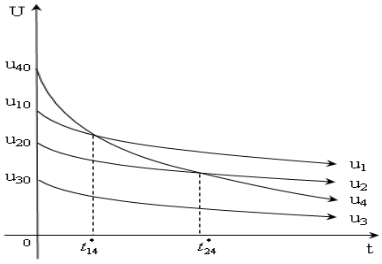
Where $P_1 = \dfrac{u_{30}\left(\lambda_3 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_3 - \lambda_4}$, $P_2 = \dfrac{u_{30}\left(\lambda_4 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_4 - \lambda_3}$

Where u_{10}, u_{20}, u_{30}, u_{40} are the initial values of u_1, u_2, u_3, u_4 respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t)$, $u_{20}(t)$, $u_{30}(t)$, $u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

The solutions are illustrated in figures.

Case (i): If $u_{30} < u_{20} < u_{10} < u_{40}$ and $a_3 < a_1 < A_2 < a_4$



In this case initially S_4 dominates the Prey (S_1) and the Predator (S_2) till the time instant t^*_{14}, t^*_{24} respectively and thereafter the dominance is reversed. And u_1, u_2, u_3, u_4 are converging asymptotically to the equilibrium point. Hence the equilibrium point is **stable**.

Case (ii): If $u_{40} < u_{10} < u_{30} < u_{20}$ and $A_2 < a_3 < a_1 < a_4$

In this case initially the Predator (S_2) dominates the Prey (S_1) till the time instant t^*_{12} and thereafter the dominance is reversed. Also the host (S_3) of S_2 dominates the Prey (S_1) and S_4 till the time instant t^*_{13}, t^*_{43} respectively and the dominance gets reversed thereafter.

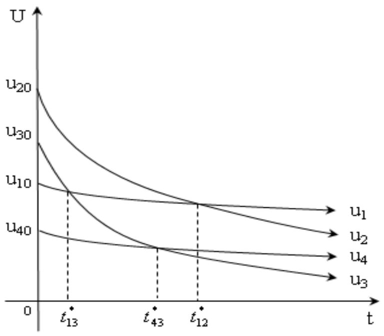


Fig. 2

Case (B): If $w_1 > 0$ [i.e. $a_1 > a_{12}\overline{N_2}$]

Here the root w_1 is positive and the other three roots are negative.

Hence the equilibrium state is **unstable** and the solutions in this case are same as in Case (A).
The solutions are illustrated in figures.

Case (i): If $u_{30} < u_{40} < u_{20} < u_{10}$ and $a_1 < a_4 < A_2 < a_3$

In this case initially S_4 dominates the host (S_3) of S_2, the Prey (S_1) and the Predator (S_2) in natural growth rate as well as in its initial population strength.

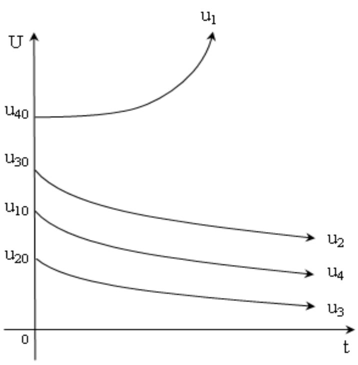
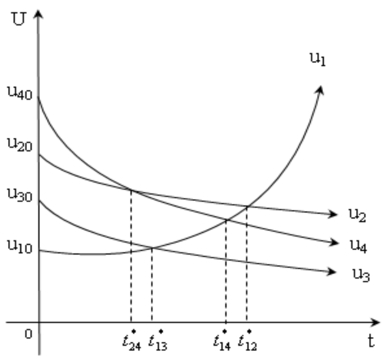


Fig. 3

Case (ii): If $u_{10} < u_{30} < u_{20} < u_{40}$ and $a_1 < a_3 < a_4 < A_2$

In this case initially S_4 dominates the Predator (S_2) and the Prey (S_1) till the time instant t^*_{24}, t^*_{14} respectively and thereafter the dominance is reversed. Also the host (S_3) of S_2 dominates the Prey (S_1) till the time instant t^*_{13} and the dominance gets reversed thereafter. And the

Predator (S_2) dominates the Prey (S_1) till the time instant t^*_{13} and the dominance gets reversed thereafter.



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B. Stability of the Equilibrium State E_{13}:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_1u_1 - a_{12}\overline{N_1}u_2 \quad \dots (4.2.1) \qquad \frac{du_2}{dt} = g_2u_2 \quad \dots (4.2.2)$$

$$\frac{du_3}{dt} = -a_{33}\overline{N_3}u_3 + a_{34}\overline{N_3}u_4 \quad \dots (4.2.3) \qquad \frac{du_4}{dt} = a_{43}\overline{N_4}u_3 - a_{44}\overline{N_4}u_4 \quad \dots (4.2.4)$$

Here $g_2 = a_2 + a_{21}\overline{N_1} + a_{23}\overline{N_3}$ ... (4.2.5)

The characteristic equation of which is

$$(\lambda_1 + a_1)(\lambda - g_2)\left[\lambda^2 + (a_{33}\overline{N_3} + a_{44}\overline{N_4})\lambda + (a_{33}a_{44} - a_{34}a_{43})\overline{N_3}\,\overline{N_4}\right] = 0 \quad \dots (4.2.6)$$

The characteristic roots of (4.2.6) are

$$\lambda = -a_1, \lambda = g_2, \lambda = \frac{-(a_{33}\bar{N}_3 + a_{44}\bar{N}_4) \pm \sqrt{(a_{33}\bar{N}_3 - a_{44}\bar{N}_4)^2 + 4a_{34}a_{43}\bar{N}_3\bar{N}_4}}{2}$$

One root of the characteristic equation $\lambda = r_2$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

$$u_1 = \left(u_{10} + \frac{a_{12}a_1u_{20}}{a_{11}(g_2 + a_1)}\right)e^{-a_1t} - \frac{a_{12}a_1u_{20}}{a_{11}(g_2 + a_1)}e^{g_2t} \quad \dots (4.6.7)$$

$$u_2 = u_{20}e^{g_2t} \quad \dots (4.6.8)$$

$$u_3 = p_1e^{\lambda_3 t} + p_2e^{\lambda_4 t} \quad \dots (4.6.9)$$

$$u_4 = \left[\frac{u_{40}\left(\lambda_3 + a_{33}\bar{N}_3\right) + u_{30}a_{43}\bar{N}_4}{\lambda_3 - \lambda_4}\right]e^{\lambda_3 t} + \left[\frac{u_{40}\left(\lambda_4 + a_{33}\bar{N}_3\right) + u_{30}a_{43}\bar{N}_4}{\lambda_4 - \lambda_3}\right]e^{\lambda_4 t} \quad \dots (4.6.10)$$

Where

$$p_1 = \frac{u_{30}\left(\lambda_3 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_3 - \lambda_4}, \quad p_2 = \frac{u_{30}\left(\lambda_4 + a_{44}\bar{N}_4\right) + u_{40}a_{34}\bar{N}_3}{\lambda_4 - \lambda_3}$$

The solutions are illustrated in figures.

Case (i): If $u_{40} < u_{20} < u_{10} < u_{30}$ and $a_4 < a_3 < a_1 < g_2$

In this case initially the host (S_3) of S_2 dominates the Prey (S_1) till the time instant t^*_{23} and the dominance gets reversed thereafter. Also the Prey (S_1) dominates the Predator (S_2) till the time instant t^*_{21} and thereafter the dominance is reversed.

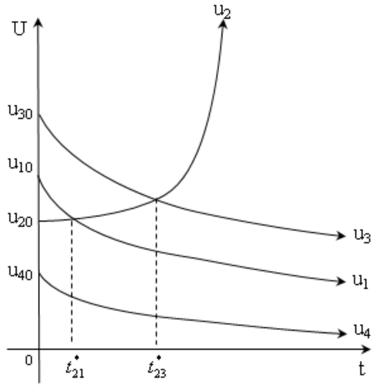

Fig. 5

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Case (ii): If $u_{10} < u_{20} < u_{40} < u_{30}$ and $g_2 < a_1 < a_4 < a_3$

In this case initially the host (S_3) of S_2 dominates the Predator (S_2) till the time instant t^*_{23} and thereafter the dominance is reversed. Also S_4 dominates the Predator (S_2) till the time instant t^*_{24} and the dominance gets reversed thereafter.

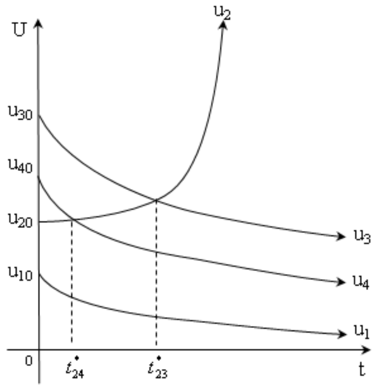

Fig. 6

C. Stability of the Equilibrium State E_{14}:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_{11}\overline{N}_1 u_1 - a_{12}\overline{N}_1 u_2 \qquad \text{… (4.3.1)}$$

$$\frac{du_2}{dt} = (a_2 - 2a_{22}\overline{N}_2 + a_{21}\overline{N}_1)u_2 + a_{21}\overline{N}_2 u_1 + a_{23}\overline{N}_2 u_3 \qquad \text{... (4.3.2)}$$

$$\frac{du_3}{dt} = l_3 u_3 \qquad \text{… (4.3.3)} \qquad \frac{du_4}{dt} = -a_4 u_4 + a_{43}\frac{a_4}{a_{44}} u_3 \quad \text{... (4.3.4)}$$

Here $l_3 = a_3 + a_{34}\dfrac{a_4}{a_{44}}$ ... (4.3.5)

The characteristic equation of which is

$$(\lambda - l_3)(\lambda + a_4)\left[\lambda^2 - (M_3 - a_{11}\overline{N}_1)\lambda + [(-a_{11}\overline{N}_1)(M_3) + a_{12}a_{21}\overline{N}_1\overline{N}_2\right] = 0 \qquad \text{... (4.3.6)}$$

The characteristic roots of (4.3.6) are

$$\lambda = l_3, \lambda = -a_4,$$

$$\lambda = \frac{(M_3 - a_{11}\overline{N}_1) \pm \sqrt{(M_3 - a_{11}\overline{N}_1)^2 - [a_{12}a_{21}\overline{N}_1\overline{N}_2 - M_3 a_{11}\overline{N}_1]}}{2}$$

Where $M_3 = a_2 - 2a_{22}\overline{N}_2 + a_{21}\overline{N}_1$

One root of the characteristic equation $\lambda = l_3$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

$$u_1 = \left\{\frac{(u_{10} + u_{20})a_{12}\overline{N}_1 - \phi_3(\lambda_2 - l_3)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_1 t}$$
$$+ \left\{\frac{(u_{10} - \phi_3)(\lambda_2 - \lambda_1) - (u_{10} + u_{20})a_{12}\overline{N}_1 + \phi_3(\lambda_2 - l_3)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_2 t} + \phi_3 e^{l_3 t} \qquad \text{… (4.3.7)}$$

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$$u_2 = \left\{ \frac{(u_{10}+u_{20})a_{12}\overline{N_1} - \phi_3(\lambda_2 - l_3)}{\lambda_2 - \lambda_1} \right\} e^{\lambda_1 t}\xi_1 + \left\{ \frac{(u_{10}-\phi_3)(\lambda_2-\lambda_1) - (u_{10}+u_{20})a_{12}\overline{N_1} + \phi_3(\lambda_2 - l_3)}{\lambda_2 - \lambda_1} \right\} e^{\lambda_2 t}\xi_2 + \phi_4 e^{l_3 t} \quad \ldots (4.3.8)$$

$$u_3 = u_{30} e^{l_3 t} \quad \ldots (4.3.9)$$

$$u_4 = \left[u_{40} - \frac{a_4 a_{43} u_{30}}{a_{44}(l_3 + a_4)} \right] e^{-a_4 t} + \frac{a_4 a_{43} u_{30}}{a_{44}(l_3+a_4)} e^{l_3 t} \quad \ldots (4.3.10)$$

Where

$$\phi_3 = \frac{\beta_4}{l_3^{\,2} + \psi_2 l_3 + \beta_3}, \psi_2 = a_{11}\overline{N_1} - a_{21}\overline{N_1} + 2a_{22}\overline{N_2} - a_2, \phi_4 = \frac{\phi_3(l_3+p_3)}{a_{12}\overline{N_1}}$$

$$\beta_3 = (2a_{11}a_{22}\overline{N_2} - a_2 a_{11} - a_{11}a_{21}\overline{N_1} - a_{12}a_{21}\overline{N_2})\overline{N_1}, \beta_4 = -a_{12}a_{23}u_{30}\overline{N_1}\,\overline{N_2}$$

$$\xi_1 = \frac{-(\lambda_1 + p_3)}{a_{12}\overline{N_1}}, \xi_2 = \frac{-(\lambda_2+p_3)}{a_{12}\overline{N_1}}, p_3 = a_{11}\overline{N_1}$$

The solutions are illustrated in figures.

Case (i): If $u_{30} < u_{20} < u_{40} < u_{10}$ and $l_3 < a_1 < a_2 < a_4$

In this case initially the Prey (S_1) dominates S_4 and the host (S_3) of S_2 till the time instant t^*_{41}, t^*_{31} respectively and thereafter the dominance is reversed. Also S_4 dominates the host (S_3) of S_2 till the time instant t^*_{34} and the dominance gets reversed thereafter. Similarly the Predator (S_2) dominates the host (S_3) of S_2 till the time instant t^*_{32} and thereafter the dominance is reversed.

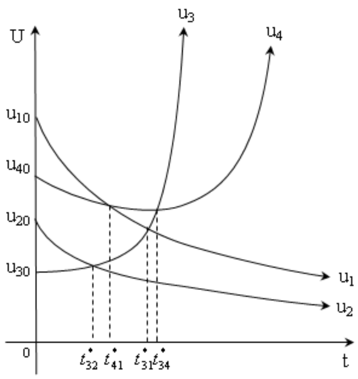

Fig. 7

Case (ii): If $u_{10} < u_{30} < u_{40} < u_{20}$ and $a_4 < a_1 < l_3 < a_2$

In this case initially the Predator (S_2) dominates S_4, the host (S_3) of S_2 and the Prey (S_1) till the time instant t^*_{42}, t^*_{32}, t^*_{12} respectively and thereafter the dominance is reversed. Also S_4 dominates the host (S_3) of S_2 till the time instant t^*_{34} and the dominance gets reversed thereafter.

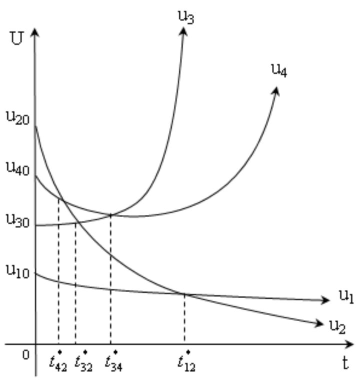

Fig. 8

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D. Stability of the Equilibrium State E_{15}:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_{11}\overline{N}_1 u_1 - a_{12}\overline{N}_1 u_2 \qquad \ldots (4.4.1)$$

$$\frac{du_2}{dt} = -a_{22}\overline{N}_2 u_2 + a_{21}\overline{N}_2 u_1 + a_{23}\overline{N}_2 u_3 \qquad \ldots (4.4.2)$$

$$\frac{du_3}{dt} = -a_3 u_3 + a_{34}\overline{N}_3 u_4 \qquad \ldots (4.4.3) \qquad \frac{du_4}{dt} = n_4 u_4 \qquad \ldots (4.4.4)$$

Here $n_4 = a_4 + a_{43}\dfrac{a_3}{a_{33}}$ ... (4.4.5)

The characteristic equation of which is

$$\left[\lambda^2 + (a_{11}\overline{N}_1 + a_{22}\overline{N}_2)\lambda + (a_{11}a_{22} + a_{12}a_{21})\overline{N}_1\overline{N}_2\right](\lambda - n_4)(\lambda + a_3) = 0 \qquad \ldots (4.4.6)$$

The characteristic roots of (4.15.6) are

$$\lambda = \frac{-(a_{11}\bar{N}_1 + a_{22}\bar{N}_2) \pm \sqrt{(a_{11}\bar{N}_1 - a_{22}\bar{N}_2)^2 - 4a_{12}a_{21}\bar{N}_1\bar{N}_2}}{2}, \lambda = n_4, \lambda = -a_3$$

One root of the characteristic equation $\lambda = n_4$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

$$u_1 = \left\{\frac{(u_{10} + u_{20} - w_3^* - w_4^*)a_{12}\overline{N}_1 - (\lambda_2 + P_3)(w_3 + w_4)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_1 t}$$
$$+ \left\{\frac{[u_{10} - (w_3 + w_4)](\lambda_2 - \lambda_1) + (u_{10} + u_{20} - w_3^* - w_4^*)a_{12}\overline{N}_1 - (\lambda_2 + P_3)(w_3 + w_4)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_2 t}$$
$$+ w_3 e^{-a_3 t} + w_4 e^{n_4 t} \qquad \ldots (4.4.7)$$

$$u_2 = \left\{\frac{(u_{10} + u_{20} - w_3^* - w_4^*)a_{12}\overline{N}_1 - (\lambda_2 + P_3)(w_3 + w_4)}{\lambda_2 - \lambda_1}\right\}\xi_1 e^{\lambda_1 t}$$
$$+ \left\{\frac{[u_{10} - (w_3 + w_4)](\lambda_2 - \lambda_1) + (u_{10} + u_{20} - w_3^* - w_4^*)a_{12}\overline{N}_1 - (\lambda_2 + P_3)(w_3 + w_4)}{\lambda_2 - \lambda_1}\right\}\xi_2 e^{\lambda_2 t} + w_3^* e^{-a_3 t} + w_4^* e^{n_4 t} \qquad \ldots (4.4.8)$$

$$u_3 = \left[u_{30} - \frac{a_3 a_{34} u_{40}}{a_{33}(n_4 + a_3)}\right]e^{-a_3 t} + \frac{a_3 a_{34} u_{40}}{a_{33}(n_4 + a_3)}e^{n_4 t} \qquad \ldots (4.4.9)$$

$$u_4 = u_{40}e^{n_4 t} \qquad \ldots (4.4.10)$$

where

$$w_3 = \frac{(u_{30} - r_4)a_{12}a_{23}\overline{N}_1\overline{N}_2}{D^2 - \psi_1 D + \beta_5}, w_4 = \frac{a_{12}a_{23}\overline{N}_1\overline{N}_2)}{D^2 - \psi_1 D + \beta_5}, D = -a_3$$

$$w_3^* = w_3(a_3 - P_3), w_4^* = -(n_4 + P_3)\ , \ \xi_1 = \frac{-(\lambda_1 + p_3)}{a_{12}\overline{N}_1}, \xi_2 = \frac{-(\lambda_2 + p_3)}{a_{12}\overline{N}_1}, p_3 = a_{11}\overline{N}_1$$

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$$\psi_1 = a_{11}\overline{N}_1 + a_{22}\overline{N}_2,\ \beta_5 = a_{11}a_{22}\overline{N}_1\overline{N}_2 + a_{21}\overline{N}_2,\ r_4 = \frac{a_3 a_{34} u_{40}}{a_{33}(n_4 + a_3)}$$

The solutions are illustrated in figures.

Case (i): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $n_4 < a_3 < a_2 < n_1$

In this case initially the Predator (S_2) dominates S_4 and the host (S_3) of S_2 till the time instant t^*_{42}, t^*_{32} respectively and thereafter the dominance is reversed. Also the Prey (S_1) dominates S_4 and the host (S_3) of S_2 till the time instant t^*_{41}, t^*_{31} respectively and the dominance gets reversed thereafter.

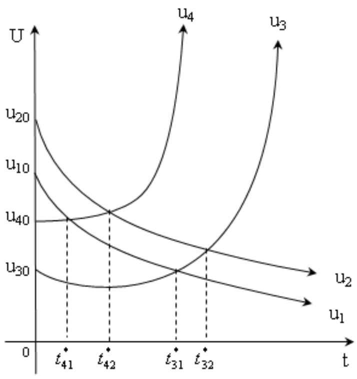

Fig. 9

Case (ii): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $n_4 < a_3 < a_2 < n_1$

In this case initially the host (S_3) of S_2 dominates S_4 till the time instant t^*_{43} and thereafter the dominance is reversed. Also the Predator (S_2) dominates S_4 till the time instant t^*_{42} and the dominance gets reversed thereafter. Similarly the Prey (S_1) dominates S_4 till the time instant t^*_{41} and thereafter the dominance is reversed.

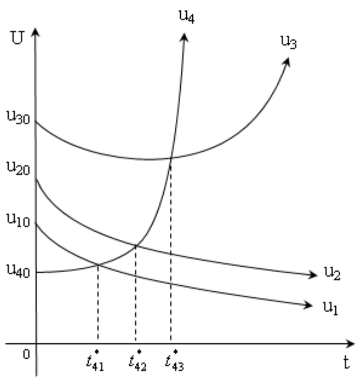

Fig. 10

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