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GENERATED SETS OF THE COMPLETE SEMIGROUP BINARI RELATIONS DEFINED BY SEMILATTICES OF THE CLASS $\Sigma_1(X,3)$

Yasha Diasamidze [1] Omar Givradze [2] Alexander Bakuridze[3]
Faculty of Physics, Mathematics and Computer Sciences, Department of Mathematics,
Shota Rustaveli Batumi State University, 35 Ninoshvili St., Batumi 6010,Georgia

Abstract: ***In this article, we study generated sets of the complete semigroups defined by*** $X-$ ***semilattices unions of the class*** $\Sigma_1(X,3)$.

Key words: **Semigroup, semi lattice, binary relation.**

I. INTRODUCTION

1.1. Let X be an arbitrary nonempty set, D is an $X-$semi lattice of unions which closed with respect to the set-theoretic union of elements from D, f be an arbitrary mapping of the set X in the set D. To each mapping f we put into correspondence a binary relation α_f on the set X that satisfies the condition

$$\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x)).$$

The set of all such α_f ($f: X\to D$) is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semi group with respect to the operation of multiplication of binary relations, which is called a complete semi group of binary relations defined by an $X-$semi lattice of unions D.

We denote by $\varnothing$ an empty binary relation or an empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Further, let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $\breve{D}=\bigcup_{Y\in D} Y$ and $T\in D$. We denote by the symbols $y\alpha$, $Y\alpha$, $V(D,\alpha)$, X^* and $V(X^*,\alpha)$ the following sets:

$$y\alpha=\{x\in X \mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y} y\alpha,\ V(D,\alpha)=\{Y\alpha \mid Y\in D\},$$
$$X^*=\{Y \mid \varnothing\neq Y\subseteq X\},\ \ V(X^*,\alpha)=\{Y\alpha \mid \varnothing\neq Y\subseteq X\},$$
$$D_T=\{Z\in D \mid T\subseteq Z\}.\ Y_T^{\alpha}=\{y\in X \mid y\alpha=T\}$$

It is well know the following statement:

Theorem 1.1. *Let* $D=\{\breve{D},Z_1,Z_2,...,Z_{m-1}\}$ *be some finite* $X-$*semi lattice of unions and* $C(D)=\{P_0,P_1,P_2,...,P_{m-1}\}$ *be the family of sets of pair wise nonintersecting subsets of the set* X *(the set* $\varnothing$ *can be repeat several time). If* φ *is a mapping of the semi lattice* D *on the family of sets* $C(D)$ *which satisfies the condition*

$$\varphi=\begin{pmatrix}\breve{D} & Z_1 & Z_2 & ... & Z_{m-1}\\ P_0 & P_1 & P_2 & ... & P_{m-1}\end{pmatrix}$$

and $\hat{D}_Z=D\setminus D_Z$, *then the following equalities are valid:*

$$\breve{D}=P_0\cup P_1\cup P_2\cup ...\cup P_{m-1},\qquad \text{... (1.1)}$$
$$Z_i=P_0\cup\bigcup_{T\in\hat{D}_{Z_i}}\varphi(T).$$

In the sequel these equalities will be called formal.

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It is proved that if the elements of the semi lattice D are represented in the form (1.1), then among the parameters P_i $(0<i\le m-1)$ there exist such parameters that cannot be empty sets for D. Such sets P_i are called basis sources, whereas sets P_j $(0\le j\le m-1)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1], chapter 11).

Let $P_0, P_1, P_2, ..., P_{m-1}$ be parameters in the formal equalities and β be any binary relation of the semi group $B_X(D)$ and

$$\overline{\beta}=\bigcup_{i=0}^{m-1}\left(P_i\times\bigcup_{t\in P_i}t\beta\right)\cup\bigcup_{t'\in X\setminus\breve{D}}\left(\{t'\}\times\overline{\beta}_2(t')\right), \qquad \dots(1.2)$$

where $\overline{\beta}_2$ is any mapping of the set $X\setminus\breve{D}$ in the set D. Then the representation of the binary relation β of the form $\overline{\beta}$ will be called subquasinormal.

If $\overline{\beta}$ are the subquasinormal representations of the binary relation β, then for the binary relations $\overline{\beta}$ the following statements are true:

$a)$ $\overline{\beta}\in B_X(D)$;

$b)$ $\bigcup_{i=0}^{m-1}\left(P_i\times\bigcup_{t\in P_i}t\beta\right)\subseteq\beta$ and $\beta\subseteq\overline{\beta}$ for some mapping $\overline{\beta}_2$ of the set $X\setminus\breve{D}$ in the set D.

$c)$ the subquasinormal representation of the binary relation β is quasinormal;

$d)$ if $\overline{\beta}_1=\begin{pmatrix}P_0 & P_1 & \dots & P_{m-1}\\ P_0\beta & P_1\beta & \dots & P_{m-1}\beta\end{pmatrix}$, then $\overline{\beta}_1$ is a mapping of the family of sets $C(D)$ in the set $D\cup\{\varnothing\}$.

Remark, that if P_j $(0\le j\le m-1)$ is such completeness sources, that $P_j=\varnothing$, then the equality $P_j\overline{\beta}=\varnothing$ always is hold. There also exists such a basic sources P_i $(0\le i\le m-1)$ for which $\bigcup_{t\in P_i}t\beta=\varnothing$, i.e. $P_i\overline{\beta}=\varnothing$.

Definition 1.1. In the sequel, the elements $\overline{\beta}_1$ and $\overline{\beta}_2$ will be called normal and complement mappings for the binary relation $\overline{\beta}\in B_X(D)$.

Theorem 1.2. *Let X is finite a set and $\alpha,\beta\in B_X(D)$, then for any subquasinormal representation $\overline{\beta}$ of a binary relation β the equality $\alpha\circ\beta=\alpha\circ\overline{\beta}$ is hold (see [2], Proposition 2).*

Proof. Let $x(\alpha\circ\beta)y$ for some $x\in X$ and $y\in\breve{D}$. Then $x\alpha z\beta y$ for some $z\in\breve{D}$ since $x\alpha z$. So, we have $z\overline{\beta}y$ by definition subquasinormal representation $\overline{\beta}$ of a binary relation β and $z,y\in\breve{D}$. Thus the condition $x\alpha z\overline{\beta}y$ is hold, i.e. $\alpha\circ\beta\subseteq\alpha\circ\overline{\beta}$.

In the other hand, if $x'\alpha z'\overline{\beta}y'$ for some $x',z',y'\in X$, then $z',y'\in\breve{D}$ since $\alpha,\overline{\beta}\in B_X(D)$. From the condition $z'\in\breve{D}$ and the formal equalities follows that $z'\in P_k$ for some $0\le k\le m-1$, i.e. $z'\left(\bigcup_{i=0}^{m-1}\left(P_i\times\bigcup_{t\in P_i}t\beta\right)\right)y'$. Of the last condition and from the condition $\bigcup_{i=0}^{m-1}\left(P_i\times\bigcup_{t\in P_i}t\beta\right)\subseteq\beta$ we obtain that the conditions $z'\beta y'$ and $x'\alpha z'\beta y'$ are hold. So, we have that $\alpha\circ\overline{\beta}\subseteq\alpha\circ\beta$.

Therefore the equality $\alpha\circ\beta=\alpha\circ\overline{\beta}$ is true.

Theorem 1.2 is proved.

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Theorem 1.3. *Let $\tilde{B}$ be any generating set of the semi group $B_X(D)$. If for some α and δ of the set $\tilde{B}$ and subquasinormal representation $\overline{\beta} \in B_X(D)$ of a binary relation $\beta \in \tilde{B}$ the inequality $\alpha \neq \delta \circ \overline{\beta}$ is hold, then the condition $\alpha \neq \delta \circ \beta$ is also true.*

Proof. If $\alpha = \delta \circ \beta$ for some $\alpha, \delta, \beta \in \tilde{B}$, then from the theorem 1.2 follows, that $\alpha = \delta \circ \beta = \delta \circ \overline{\beta}_1$ for some $\overline{\beta}_1 \in B_X(D)$ and $\overline{\beta}$ is some subquasinormal representation of a binary relation β. But equality $\alpha = \delta \circ \overline{\beta}_1$ contradict the condition $\alpha \neq \delta \circ \overline{\beta}$ for any subquasinormal representations $\overline{\beta} \in B_X(D)$ of a binary relation β. Thus, we have that the representation of a binary relation α of the form $\alpha \neq \delta \circ \beta$ is true.
Theorem 1.3 is proved.

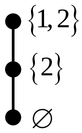

Fig. 1.1

Example 1.1. Let $X = \{1,2,3,4,5\}$, $D = \{\varnothing, \{2\}, \{1,2\}\}$, then $P_0 = \varnothing$, $P_1 = \{1\}$, $P_2 = \{2\}$. If $\beta = \{(2,1),(2,2),(3,1),(4,1),(4,2),(5,1)\}$, then $\beta \in B_X(D)$, $\overline{\beta}_1 = \begin{pmatrix} \varnothing & P_1 & P_2 \\ \varnothing & \varnothing & \{1,2\} \end{pmatrix}$, $\overline{\beta}_2 = \begin{pmatrix} 3 & 4 & 5 \\ \{1\} & \{1,2\} & \{1\} \end{pmatrix}$ and subquasinormal representation of a binary relation $\overline{\beta}$ has a form

$$\overline{\beta} = (P_0 \times \varnothing) \cup (P_1 \times \varnothing) \cup (P_2 \times \{1,2\}) \cup (\{3\} \times \{1\}) \cup (\{4\} \times \{1,2\}) \cup (\{5\} \times \{1\})$$

where P_1, P_2 are basic sources and P_0 is completeness sources.

Definition 1.2. *We say that an element α of the semi group $B_X(D)$ is external if $\alpha \neq \delta \circ \beta$ for all $\delta, \beta \in B_X(D) \setminus \{\alpha\}$ (see [1], Definition 1.15.1).*

It is well know, that if B is all external elements of the semi group $B_X(D)$ and B' be any generated set for the $B_X(D)$, then $B \subseteq B'$ (see [1], Lemma 1.15.1).

2.1. Let $\Sigma_1(X,3)$ be a class of all $X-$semilattices of unions whose every element is isomorphic to an $X-$semi lattice of unions $D = \{Z_2, Z_1, \breve{D}\}$, which satisfies the condition $Z_2 \subset Z_1 \subset \breve{D}$ (see Figure 2.1):

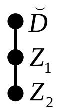

Fig. 2.1

Let $C(D) = \{P_0, P_1, P_2\}$ is a family sets, where P_0, P_1, P_2 are pairwise disjoint subsets of the set X and $\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 \\ P_0 & P_1 & P_2 \end{pmatrix}$ is a mapping of the semilattice D onto the family sets $C(D)$. Then for the formal equalities of the semilattice D we have a form:

$$\begin{aligned} \breve{D} &= P_0 \cup P_1 \cup P_2, \\ Z_1 &= P_0 \cup P_2, \\ Z_2 &= P_0, \end{aligned} \qquad \ldots(2.1)$$

Here the elements P_1, P_2 are basis sources, the element P_0 is sources of completeness of the semilattice D. Therefore $|X| \geq 2$ since $|P_1| \geq 1$ and $|P_2| \geq 1$.

It is well know the following statement (see [4],).

Theorem 2.1. *Let $D = \{Z_2, Z_1, \breve{D}\} \in \Sigma_1(X,3)$ and $Z_2 \neq \varnothing$. If $E_X^{(r)}(D)$ be the set all right units of the semigroup $B_X(D)$,*

$$\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1), \; \sigma_2 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times \breve{D}),$$
$$\sigma_3 = (Z_1 \times Z_2) \cup ((X \setminus Z_1) \times \breve{D}), \; \sigma_4 = (Z_1 \times Z_1) \cup ((X \setminus Z_1) \times \breve{D})$$

and $B' = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, then $B = E_X^{(r)}(D) \cup B'$ is irreducible generated set for the semigroup $B_X(D)$.

In the sequel, we will be assumption, that $Z_2 = \varnothing$.

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Lemma 2.1. *Let* $D=\{\varnothing,Z_1,\breve{D}\}\in\Sigma_1(X,3)$ *and* $B=\{\alpha\in B_X(D)\,|\,V(X^*,\alpha)=D\}$. *Then the following statements are true:*

a) $B\neq\varnothing$ *if and only if, when* $|X|\geq 3$;

b) $P_0=\cap D=\varnothing$, $P_1=\breve{D}\setminus Z_1$ *and* $P_2=Z_1$;

c) *If* $\alpha=\delta\circ\beta$, *for some* $\alpha\in B$, $\delta,\beta\in B_X(D)$, *then* $V(D,\beta)=D$;

d) *if* $|X|\geq 3$, *then B is a set external elements of the semigroup* $B_X(D)$.

Proof. Let $D=\{\varnothing,Z_1,\breve{D}\}\in\Sigma_1(X,3)$,

1) If $B\neq\varnothing$ and $\alpha\in B$ for some $\alpha\in B_X(D)$, then, there exists quasinormal representations of a binary relation α of the form

$$\alpha=(Y_2^\alpha\times\varnothing)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D}),$$

Where $|Y_i^\alpha|\geq 1$ for all $i=0,1,2$ (if $Y_j^\delta=\varnothing$ for some j $(0\leq j\leq 2)$, then $V(X^*,\alpha)\neq D$). So, the inequality $|X|\geq 3$ is true. Of this we obtain, that $B=\varnothing$, if $|X|=2$.

The statement $a)$ of the Lemma 2.1 is proved.

2) By assumption $Z_2=\varnothing$, then by definition of the set P_0 we obtain, that $P_0=\cap D=\varnothing$. Now, from the formal equality (2.1) follows that $P_2=Z_1$ and $P_1=\breve{D}\setminus P_2=\breve{D}\setminus Z_1$ since $P_2\cap P_1=\varnothing$.

The statement $b)$ of the Lemma 2.1 is proved.

3) Let $\alpha=\delta\circ\beta$, for some $\alpha\in B$, $\delta,\beta\in B_X(D)$. Then $D=V(X^*,\alpha)\subseteq V(D,\beta)$ (see [1], Theorem 4.1.1). So, $D=V(D,\beta)$ since the inclusion $V(D,\beta)\subseteq D$ for any semilattice D always is hold.

The statement $c)$ of the Lemma 2.1 is proved.

4) Now, let $\alpha=\delta\circ\beta$ for some $\alpha\in B$ and $\delta,\beta\in B_X(D)\setminus\{\alpha\}$, then quasinormal representation of a binary relations α and δ has a form

$\alpha=(Y_2^\alpha\times\varnothing)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$ and $\delta=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times Z_1)\cup(Y_0^\delta\times\breve{D})$,

Where $Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$, i.e. $V(X^*,\alpha)=D$. By Theorem 1.2 follows that $\alpha=\delta\circ\beta=\delta\circ\overline{\beta}$, where $\overline{\beta}$ is subquasinormal representation of a binary relation β. It is easy to see, that

$$\alpha=\delta\circ\overline{\beta}=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times Z_1\overline{\beta})\cup(Y_0^\delta\times\breve{D}\overline{\beta}). \qquad \ldots(2.2)$$

For the sets X and $\breve{D}$ we consider the following cases:

a′) $X=\breve{D}$. Then from the equality (1.2) follows that $\overline{\beta}_2$ is empty mapping since $X\setminus\breve{D}=\varnothing$. So, there exists only two subquasinormal representations $\overline{\beta}$ of a binary relation β for which $V(D,\beta)=D$ (see statement $c)$ of the Lemma 2.1) and $\overline{\beta}=\beta$:

$$\overline{\beta}=(\varnothing\times\varnothing)\cup((\breve{D}\setminus Z_1)\times Z_1)\cup(Z_1\times\breve{D}) \text{ or } \overline{\beta}=(\varnothing\times\varnothing)\cup(Z_1\times Z_1)\cup((\breve{D}\setminus Z_1)\times\breve{D}),$$

where $\overline{\beta}\in B_X(D)$.

If $\overline{\beta}=(\varnothing\times\varnothing)\cup((\breve{D}\setminus Z_1)\times Z_1)\cup(Z_1\times\breve{D})$, then

$$\delta\circ\overline{\beta}=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times Z_1\overline{\beta})\cup(Y_0^\delta\times\breve{D}\overline{\beta})=$$
$$=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times\breve{D})\cup(Y_0^\delta\times\breve{D})=(Y_2^\delta\times\varnothing)\cup((Y_1^\delta\cup Y_0^\delta)\times\breve{D})\notin B$$

since $V(X^*,\delta\circ\overline{\beta})\subseteq\{\varnothing,\breve{D}\}\neq D$. So, we have that $\alpha\neq\delta\circ\overline{\beta}$.

If $\overline{\beta}=(\varnothing\times\varnothing)\cup(Z_1\times Z_1)\cup((\breve{D}\setminus Z_1)\times\breve{D})$, then

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$$\delta\circ\bar{\beta}=\left(Y_2^{\alpha}\times\varnothing\right)\cup\left(Y_1^{\alpha}\times Z_1\right)\cup\left(Y_0^{\alpha}\times\breve{D}\right)=$$
$$=\left(Y_2^{\delta}\times\varnothing\right)\cup\left(Y_1^{\delta}\times Z_1\bar{\beta}\right)\cup\left(Y_0^{\delta}\times\breve{D}\bar{\beta}\right)=$$
$$=\left(Y_2^{\delta}\times\varnothing\right)\cup\left(Y_1^{\delta}\times Z_1\right)\cup\left(Y_0^{\delta}\times\breve{D}\right).$$

So, $Y_2^{\delta}=Y_2^{\alpha}$, $Y_1^{\delta}=Y_1^{\alpha}$, $Y_0^{\delta}=Y_0^{\alpha}$. Of this follows that $\alpha=\delta$. But, the equality $\alpha=\delta$ contradict the condition $\delta\in B_X(D)\setminus\{\alpha\}$. So, we have that $\alpha\neq\delta\circ\bar{\beta}$.

In the sequel we will be assumption, that $X\neq\breve{D}$.

b′) Let $\left|X\setminus\breve{D}\right|\geq 1$. By preposition we have $P_0=\varnothing$. In this case

$$\bar{\beta}_1^1=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & \varnothing & \varnothing\end{pmatrix},\ \bar{\beta}_1^2=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & \varnothing & Z_1\end{pmatrix},\ \bar{\beta}_1^3=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & Z_1 & \varnothing\end{pmatrix},$$
$$\beta_1^4=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & Z_1 & Z_1\end{pmatrix},\ \bar{\beta}_1^5=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & Z_1 & \breve{D}\end{pmatrix},\ \bar{\beta}_1^6=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & \varnothing & \breve{D}\end{pmatrix},$$
$$\bar{\beta}_1^7=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & \breve{D} & \breve{D}\end{pmatrix},\ \bar{\beta}_1^8=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & \breve{D} & \varnothing\end{pmatrix},\ \bar{\beta}_1^9=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & \breve{D} & Z_1\end{pmatrix}.$$

are all mappings of the set $C(D)=\{\varnothing,P_1,P_2\}$ (see statement $b)$ of the Lemma 2.1) in the semilattice D satisfying the condition $\bar{\beta}_1^i(P_0)=\varnothing$ $(i=1,2,...,8,9)$.

Let $\beta\in B_X(D)$ and $\bar{\beta}$ is such subquasinormal representation of a binary relation β for which β_1^i $(i=1,2,...,8,9)$ is normal mapping for the binary relation $\bar{\beta}$.

For a binary relation $\bar{\beta}$ we consider the following cases:

1) If $\bar{\beta}_1^1=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & \varnothing & \varnothing\end{pmatrix}$, and $\bar{\beta}_2^1$ be any mapping of the set $X\setminus\breve{D}$ in the set $D\setminus\{\varnothing\}=\{Z_1,\breve{D}\}$. So, if

$$\bar{\beta}=\left(\breve{D}\times\varnothing\right)\cup\bigcup_{t'\in X\setminus\breve{D}}\left(\{t'\}\times\bar{\beta}_2^1(t')\right),\qquad\ldots(2.3)$$

then $\bar{\beta}\in B_X(D)$. From the equalities (2.2) and (2.3) we obtain that:

$$Z_1\bar{\beta}=\varnothing,\ \breve{D}\bar{\beta}=\varnothing,$$
$$\delta\circ\bar{\beta}=\left(Y_2^{\delta}\times\varnothing\right)\cup\left(Y_1^{\delta}\times Z_1\bar{\beta}\right)\cup\left(Y_0^{\delta}\times\breve{D}\bar{\beta}\right)=$$
$$=\left(Y_2^{\delta}\times\varnothing\right)\cup\left(Y_1^{\delta}\times\varnothing\right)\cup\left(Y_0^{\delta}\times\varnothing\right)=X\times\varnothing=\varnothing\notin B$$

since $V\left(X^*,\delta\circ\bar{\beta}\right)\subseteq\{\varnothing\}\neq D$. So, we have that $\alpha\neq\delta\circ\bar{\beta}$.

2) If $\bar{\beta}_1^2=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & \varnothing & Z_1\end{pmatrix}$ and $\bar{\beta}_2^2$ be a mapping of the set $X\setminus\breve{D}$ in the set $D\setminus\{\varnothing,Z_1\}=\{\breve{D}\}$. So, if

$$\bar{\beta}=\left(\left(\breve{D}\setminus Z_1\right)\times\varnothing\right)\cup\left(Z_1\times Z_1\right)\cup\bigcup_{t'\in X\setminus\breve{D}}\left(\{t'\}\times\bar{\beta}_2^2(t')\right),\qquad\ldots(2.4)$$

then $\bar{\beta}\in B_X(D)$. From the equalities (2.2) and (2.4) follows that:

$$Z_1\bar{\beta}=Z_1,\ \breve{D}\bar{\beta}=Z_1,$$
$$\delta\circ\bar{\beta}=\left(Y_2^{\delta}\times\varnothing\right)\cup\left(Y_1^{\delta}\times Z_1\bar{\beta}\right)\cup\left(Y_0^{\delta}\times\breve{D}\bar{\beta}\right)=$$
$$=\left(Y_2^{\delta}\times\varnothing\right)\cup\left(Y_1^{\delta}\times Z_1\right)\cup\left(Y_0^{\delta}\times Z_1\right)=\left(Y_2^{\delta}\times\varnothing\right)\cup\left(\left(Y_1^{\delta}\cup Y_0^{\delta}\right)\times Z_1\right)\notin B$$

since $V\left(X^*,\delta\circ\bar{\beta}\right)\subseteq\{\varnothing,Z_1\}\neq D$. So, we have that $\alpha\neq\delta\circ\bar{\beta}$.

For the mapping $\bar{\beta}_1^3=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & Z_1 & \varnothing\end{pmatrix}$, we analogically above, may proved that $\alpha\neq\delta\circ\bar{\beta}$.

3) If $\beta_1^4=\begin{pmatrix}\varnothing & \breve{D}\setminus Z_1 & Z_1\\ \varnothing & Z_1 & Z_1\end{pmatrix}$ and $\bar{\beta}_2^4$ be a mapping of the set $X\setminus\breve{D}$ in the set $D\setminus\{\varnothing,Z_1\}=\{\breve{D}\}$. So, if

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$$\bar{\beta} = \left(\breve{D} \times Z_1\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\beta}_2^4(t')\right), \qquad \ldots(2.5)$$

then $\bar{\beta} \in B_X(D)$. From the equalities (2.2) and (2.5) we obtain:

$$\begin{aligned} & Z_1\bar{\beta} = \breve{D}\bar{\beta} = Z_1, \\ & \delta \circ \bar{\beta} = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\bar{\beta}\right) \cup \left(Y_0^\delta \times \breve{D}\bar{\beta}\right) = \\ & = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times Z_1\right) = \left(Y_2^\delta \times \varnothing\right) \cup \left(\left(Y_1^\delta \cup Y_0^\delta\right) \times Z_1\right) \notin B \end{aligned}$$

since $V\left(X^*, \delta \circ \bar{\beta}\right) \subseteq \{\varnothing, Z_1\} \neq D$. So, we have that $\alpha \neq \delta \circ \bar{\beta}$.

4) If $\bar{\beta}_1^5 \subseteq \beta$, where $\bar{\beta}_1^5 = \begin{pmatrix} \varnothing & \breve{D} \setminus Z_1 & Z_1 \\ \varnothing & Z_1 & \breve{D} \end{pmatrix}$ and $\bar{\beta}_2^5$ be a mapping of the set $X \setminus \breve{D}$ in the set D. So, if

$$\bar{\beta} = \left(\left(\breve{D} \setminus Z_1\right) \times Z_1\right) \cup \left(Z_1 \times \breve{D}\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\beta}_2^5(t')\right), \qquad \ldots(2.6)$$

then $\bar{\beta} \in B_X(D)$. From the equalities (2.2) and (2.6) we have:

$$\begin{aligned} & Z_1\bar{\beta} = \breve{D},\ \breve{D}\bar{\beta} = \breve{D}, \\ & \delta \circ \bar{\beta} = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\bar{\beta}\right) \cup \left(Y_0^\delta \times \breve{D}\bar{\beta}\right) = \\ & = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times \breve{D}\right) \cup \left(Y_0^\delta \times \breve{D}\right) = \left(Y_2^\delta \times \varnothing\right) \cup \left(\left(Y_1^\delta \cup Y_0^\delta\right) \times \breve{D}\right) \notin B \end{aligned}$$

since $V\left(X^*, \delta \circ \bar{\beta}\right) \subseteq \{\varnothing, \breve{D}\} \neq D$. So, we have that $\alpha \neq \delta \circ \bar{\beta}$.

For the mappings β are $\bar{\beta}_1^6 = \begin{pmatrix} \varnothing & \breve{D} \setminus Z_1 & Z_1 \\ \varnothing & \varnothing & \breve{D} \end{pmatrix}$, $\bar{\beta}_1^7 = \begin{pmatrix} \varnothing & \breve{D} \setminus Z_1 & Z_1 \\ \varnothing & \breve{D} & \breve{D} \end{pmatrix}$, then we analogically above may proved, that $\alpha \neq \delta \circ \bar{\beta}$.

5) If $\bar{\beta}_1^8 \subseteq \beta$, where $\bar{\beta}_1^8 = \begin{pmatrix} \varnothing & \breve{D} \setminus Z_1 & Z_1 \\ \varnothing & \breve{D} & \varnothing \end{pmatrix}$ and $\bar{\beta}_2^8$ be a mapping of the set $X \setminus \breve{D}$ in the $D \setminus \{\varnothing, \breve{D}\} = \{Z_1\}$. So, if

$$\bar{\beta} = (Z_1 \times \varnothing) \cup \left(\left(\breve{D} \setminus Z_1\right) \times \breve{D}\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\beta}_2^8(t')\right), \qquad \ldots(2.7)$$

then $\bar{\beta} \in B_X(D)$. From the equalities (2.2) and (2.7) we obtain, that:

$$\begin{aligned} & Z_1\bar{\beta} = \varnothing,\ \breve{D}\bar{\beta} = \breve{D}, \\ & \delta \circ \bar{\beta} = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\bar{\beta}\right) \cup \left(Y_0^\delta \times \breve{D}\bar{\beta}\right) = \\ & = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times \varnothing\right) \cup \left(Y_0^\delta \times \breve{D}\right) = \left(\left(Y_2^\delta \cup Y_1^\delta\right) \times \varnothing\right) \cup \left(Y_0^\delta \times \breve{D}\right) \notin B \end{aligned}$$

since $V\left(X^*, \delta \circ \bar{\beta}\right) \subseteq \{\varnothing, \breve{D}\} \neq D$. So, we have that $\alpha \neq \delta \circ \bar{\beta}$.

6) If $\bar{\beta}_1^9 \subseteq \beta$, where $\bar{\beta}_1^9 = \begin{pmatrix} \varnothing & \breve{D} \setminus Z_1 & Z_1 \\ \varnothing & \breve{D} & Z_1 \end{pmatrix}$ and $\bar{\beta}_2^9$ be a mapping of the set $X \setminus \breve{D}$ in the semilattice D. So, if

$$\bar{\beta} = (Z_1 \times Z_1) \cup \left(\left(\breve{D} \setminus Z_1\right) \times \breve{D}\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\beta}_2^9(t')\right), \qquad \ldots(2.8)$$

then $\bar{\beta} \in B_X(D)$. From the equalities (2.2) and (2.8) follows that:

$$\begin{aligned} & Z_1\bar{\beta} = Z_1,\ \breve{D}\bar{\beta} = \breve{D}, \\ & \delta \circ \bar{\beta} = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\bar{\beta}\right) \cup \left(Y_0^\delta \times \breve{D}\bar{\beta}\right) = \\ & = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times \varnothing\right) \cup \left(Y_0^\delta \times \breve{D}\right) = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right). \end{aligned}$$

So, $Y_2^\delta = Y_2^\alpha$, $Y_1^\delta = Y_1^\alpha$, $Y_0^\delta = Y_0^\alpha$. Of this follows, that $\alpha = \delta$. But, the equality $\alpha = \delta$ contradict the condition, that $\delta \in B_X(D) \setminus \{\alpha\}$. So, we have that $\alpha \neq \delta \circ \bar{\beta}$.

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Thus, we have $\alpha \neq \delta \circ \overline{\beta}$ for any subquasinormal representation of a binary relation $\beta \in B_X(D) \setminus \{\alpha\}$ since the mappings $\overline{\beta}_1^1 - \overline{\beta}_1^9$ are all mappings of the set $C(D) = \{\varnothing, P_1, P_2\}$ in the semilattice D satisfying the condition $\overline{\beta}_1^i(\varnothing) = \varnothing$ $(i = 1, 2, ..., 9)$. Of this and by Theorem 1.3 follows that $\alpha \neq \delta \circ \beta$ for all $\delta, \beta \in B_X(D) \setminus \{\alpha\}$.

So, we have that the set B (if $B \neq \varnothing$, i.e. $|X| \geq 3$) is a set external elements of the semigroup $B_X(D)$.

Lemma 2.1 is proved.

Lemma 2.2. *Let* $|X| \geq 3$ *and* $D = \{\varnothing, Z_1, \breve{D}\} \in \Sigma_1(X, 3)$, *then the following statements are true:*

a) $Z_1\beta = \varnothing$, $\breve{D}\beta = Z_1$ *for some* $\beta \in B$ *if and only if*, *when* $|X \setminus \breve{D}| \geq 1$;

b) $Z_1\beta = \breve{D}\beta = Z_1$ *for some* $\beta \in B$ *if and only if* $|X \setminus \breve{D}| \geq 1$;

c) $Z_1\beta = \breve{D}\beta = \varnothing$ *for some* $\beta \in B$ *if and only if* $|X \setminus \breve{D}| \geq 2$.

Proof. Let $Z_1\beta = \varnothing$, $\breve{D}\beta = Z_1$ for some $\beta \in B$. Then qusinormal representation of a binary relation β has e form $\beta = (Y_2^\beta \times \varnothing) \cup (Y_1^\beta \times Z_1) \cup (Y_0^\beta \times \breve{D})$, where $Y_2^\beta, Y_1^\beta, Y_0^\beta \notin \{\varnothing\}$. By preposition $\breve{D} \cap Y_0^\beta = \varnothing$ since $\breve{D}\beta = Z_1$. So, $\varnothing \neq Y_0^\beta \subseteq X \setminus \breve{D}$, i.e. $|X \setminus \breve{D}| \geq 1$.

In the other hand, if $|X \setminus \breve{D}| \geq 1$, then $\beta = (Z_1 \times \varnothing) \cup ((\breve{D} \setminus Z_1) \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D})$ is a binary relation of the set B, for which $Z_1\beta = \varnothing$ and $\breve{D}\beta = Z_1$.

The statement $a)$ of the Lemma 2.2 is proved.

Let $Z_1\beta = \breve{D}\beta = Z_1$ for some $\beta \in B$. Then qusinormal representation of a binary relation β has e form $\beta = (Y_2^\beta \times \varnothing) \cup (Y_1^\beta \times Z_1) \cup (Y_0^\beta \times \breve{D})$, where $Y_2^\beta, Y_1^\beta, Y_0^\beta \notin \{\varnothing\}$ and $Y_0^\beta \cap \breve{D} = \varnothing$. Of this follows, that $\varnothing \neq Y_0^\beta \subseteq X \setminus \breve{D}$, i.e. $|X \setminus \breve{D}| \geq 1$ since $Y_0^\beta \notin \{\varnothing\}$.

Of the other hand, if $|X \setminus \breve{D}| \geq 1$, then for the binary relation

$$\beta = ((\breve{D} \setminus Z_1) \times \varnothing) \cup (Z_1 \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D})$$

we have $\beta \in B$ and $Z_1\beta = \breve{D}\beta = Z_1$.

The statement $b)$ of the Lemma 2.2 is proved.

Let $Z_1\beta = \breve{D}\beta = \varnothing$ for some $\beta \in B$. Then qusinormal representation of a binary relation β has e form $\beta = (Y_2^\beta \times \varnothing) \cup (Y_1^\beta \times Z_1) \cup (Y_0^\beta \times \breve{D})$, where $Y_2^\beta, Y_1^\beta, Y_0^\beta \notin \{\varnothing\}$ and $t\beta = \varnothing$ for all $t \in \breve{D}$ since $\varnothing$ is smallest element of the semilattice D. So, if $Y_2^\beta = \breve{D}$, $t_1\beta = Z_1$ and $t_0\beta = \breve{D}$ for some $t_1, t_0 \in X \setminus \breve{D}$. It is easy to see, that Y_2^β, Y_1^β and Y_0^β are smallest sets for which $\beta \in B$. Of this follows that $|X \setminus \breve{D}| \geq 2$.

Of the other hand, let $|X \setminus \breve{D}| \geq 2$, i.e. $X \setminus \breve{D} \supseteq \{t_1, t_0\}$, then for the binary relation

$$\beta = (\breve{D} \times \varnothing) \cup (\{t_1\} \times Z_1) \cup ((X \setminus (\breve{D} \cup \{t_1\})) \times \breve{D})$$

we have $\beta \in B$ since $X \setminus (\breve{D} \cup \{t_1\}) \neq \varnothing$ and $Z_1\beta = \breve{D}\beta = \varnothing$.

The statement $c)$ of the Lemma 2.2 is proved.

Lemma 2.2 is proved.

In the sequel, by symbols B_1, B_2 and B_3 we denoted the following sets:

$$B_1 = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_1, \breve{D}\}\},$$
$$B_2 = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = \{\varnothing, \breve{D}\}\},$$
$$B_3 = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = \{\varnothing, Z_1\}\}.$$

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By definition of a sets B_1, B_2 and B_3 immediately follows, that

$$B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \varnothing. \qquad \ldots(2.9)$$

Lemma 2.3. *Let* $|X| \geq 3$, $D = \{\varnothing, Z_1, \breve{D}\} \in \Sigma_1(X,3)$ *and*

$$B = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\}.$$

Then the following statements are true;

a) *the elements of the set* $B_1 \cup \{X \times Z_1, X \times \breve{D}\}$ *do not generating by elements of the set* B *;*

b) *if* $|X \setminus \breve{D}| \leq 1$*, then element* $\alpha = \varnothing$ *do not generating by elements of the set* B *;*

c) *if* $X = \breve{D}$*, then elements of the set* B_3 *do not generating by elements of the set* B .

Proof. Let $\alpha = \delta \circ \beta$ for some $\alpha \in B_X(D)$ and $\delta, \beta \in B$. Then quasinormal representation of the binary relation δ has a form $\delta = (Y_2^\delta \times \varnothing) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})$, where $Y_2^\delta, Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$. In this case the following equalities are hold:

$$\alpha = \delta \circ \beta = (Y_2^\delta \times \varnothing) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta) \qquad \ldots(2.10)$$

For the binary relation α we consider the following cases:

1) If $\alpha \in B_X(D') = B_1 \cup \{X \times Z_1, X \times \breve{D}\}$, where $D' = \{Z_1, \breve{D}\}$, then quasinormal representation of the binary relation α has a form $\alpha = (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$. From the equality (2.10) we obtain that

$$(Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}) = \alpha = \delta \circ \beta = (Y_2^\delta \times \varnothing) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta).$$

But last equality impossible since $Y_2^\delta \neq \varnothing$. So, the elements of the set $B_1 \cup \{X \times Z_1, X \times \breve{D}\}$ do not generating by elements of the set B.

The statement $a)$ of the lemma 2.3 is proved.

2) Now, if $\alpha = \varnothing$, then From the equality (2.10) follows that

$$\varnothing = \alpha = \delta \circ \beta = (Y_2^\delta \times \varnothing) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta).$$

Of the last equalities follows that $Z_1\beta = \breve{D}\beta = \varnothing$. But by statement $c)$ of the lemma 2.2 the equality $Z_1\beta = \breve{D}\beta = \varnothing$ for some $\beta \in B$ is possible only if , when $|X \setminus \breve{D}| \geq 2$. So, if $|X \setminus \breve{D}| \leq 1$, then binary relation $\alpha = \varnothing$ do not generating by elements of the set B.

The statement $b)$ of the lemma 2.3 is proved.

3) Let $X = \breve{D}$ and $\alpha \in B_3$, then quasinormal representation of the binary relation α has a form $\alpha = (Y_2^\alpha \times \varnothing) \cup (Y_1^\alpha \times Z_1)$, where $Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$. Then from the equality (2.10) follows that

$$(Y_2^\alpha \times \varnothing) \cup (Y_1^\alpha \times Z_1) = \alpha = \delta \circ \beta = (Y_2^\delta \times \varnothing) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta).$$

Last equalities is possible only if $Z_1\beta = \varnothing,\ \breve{D}\beta = Z_1$ or $Z_1\beta = \breve{D}\beta = Z_1$ since $Z_1 \subset \breve{D}$.

$a')$ If $Z_1\beta = \varnothing,\ \breve{D}\beta = Z_1$, then by statement $a)$ of the Lemma 2.2 follows that $|X \setminus \breve{D}| \geq 1$ which contradict the conditions $X = \breve{D}$.

$b')$ If $Z_1\beta = \breve{D}\beta = Z_1$, then by statement $b)$ of the Lemma 2.2 follows that $|X \setminus \breve{D}| \geq 1$ which contradict the conditions $X = \breve{D}$.

So, of the conditions $a')$ and $b')$ follows that the elements of the set B_3 do not generating by elements of the set B.

The statement $c)$ of the lemma 2.3 is proved.

Lemma 2.3 is proved.

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Lemma 2.4. *Let* $|X|\ge 3$, $D=\{\varnothing,Z_1,\breve{D}\}\in\Sigma_1(X,3)$ *and*

$$B=\{\alpha\in B_X(D)\,|\,V(X^*,\alpha)=D\},\ \gamma_0=(Z_1\times\varnothing)\cup((X\setminus Z_1)\times Z_1).$$

Then the following statements are true:

a) *if* $|X\setminus\breve{D}|\ge 1$, *then elements of the set* $B_2\cup B_3$ *are generating by elements of the set* B *;*

b) *if* $X=\breve{D}$, *then the elements of the set* B_3 *are generating by elements of the set* $B\cup B_1\cup\{\gamma_0\}$ *;*

c) *if* $X=\breve{D}$, *then the elements of the set* B_2 *are generating by elements of the set* $B_1\cup\{\gamma_0\}$.

Proof. Now, let $|X\setminus\breve{D}|\ge 1$ and α be arbitrary element of the set $B_2\cup B_3$. For the binary relation α we consider the following cases.

1) $\alpha\in B_2$. Then quasinormal representation of a binary relation α has a form $\alpha=(Y_2^\alpha\times\varnothing)\cup(Y_0^\alpha\times\breve{D})$, where $Y_2^\alpha,Y_0^\alpha\notin\{\varnothing\}$.

$a')$ If $|Y_0^\alpha|\ge 1$, then $|Y_2^\alpha|\ge 2$ $(|X|\ge 3)$ (see statement $a)$ of the Lemma 2.1). In this case we suppose, that

$$\beta=(Z_1\times\varnothing)\cup((X\setminus\breve{D})\times Z_1)\cup((\breve{D}\setminus Z_1)\times\breve{D}),$$

then $\beta\in B$ since $|X\setminus\breve{D}|\ge 1$ and

$$\begin{aligned}\delta\circ\beta&=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times Z_1\beta)\cup(Y_0^\delta\times\breve{D}\beta)=\\&=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times\varnothing)\cup(Y_0^\delta\times\breve{D})=\\&=((Y_2^\delta\cup Y_1^\delta)\times\varnothing)\cup(Y_0^\delta\times\breve{D})=\alpha,\end{aligned}$$

if $Y_2^\delta\cup Y_1^\delta=Y_2^\alpha$ and $Y_0^\delta=Y_0^\alpha$ since $|Y_2^\delta|\ge 1$, $|Y_1^\delta|\ge 1$ and $|Y_0^\delta|\ge 1$.

$b')$ Let $|Y_2^\alpha|\ge 1$, then $|Y_0^\alpha|\ge 2$ $(|X|\ge 3)$. In this case we suppose, that

$$\beta=((\breve{D}\setminus Z_1)\times\varnothing)\cup((X\setminus\breve{D})\times Z_1)\cup(Z_1\times\breve{D}),$$

then $\beta\in B$ since $|X\setminus\breve{D}|\ge 1$ and

$$\begin{aligned}\delta\circ\beta&=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times Z_1\beta)\cup(Y_0^\delta\times\breve{D}\beta)=\\&=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times\breve{D})\cup(Y_0^\delta\times\breve{D})=\\&=(Y_2^\delta\times\varnothing)\cup((Y_1^\delta\cup Y_0^\delta)\times\breve{D})=\alpha,\end{aligned}$$

if $Y_2^\delta=Y_2^\alpha$ and $Y_1^\delta\cup Y_0^\delta=Y_0^\alpha$ since $|Y_2^\delta|\ge 1$, $|Y_1^\delta|\ge 1$ and $|Y_0^\delta|\ge 1$.

Therefore, the elements of the set B_2 are generating by elements of the set B.

2) $\alpha\in B_3$. Then quasinormal representation of a binary relation α has a form $\alpha=(Y_2^\alpha\times\varnothing)\cup(Y_1^\alpha\times Z_1)$, where $Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$.

$c')$ Let $|Y_1^\alpha|\ge 1$ then $|Y_2^\alpha|\ge 2$ $(|X|\ge 3)$. In this case we suppose, that

$$\beta=(Z_1\times\varnothing)\cup((\breve{D}\setminus Z_1)\times Z_1)\cup((X\setminus\breve{D})\times\breve{D}),$$

then $\beta\in B$ since $|X\setminus\breve{D}|\ge 1$ and

$$\begin{aligned}\delta\circ\beta&=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times Z_1\beta)\cup(Y_0^\delta\times\breve{D}\beta)=\\&=(Y_2^\delta\times\varnothing)\cup(Y_1^\delta\times\varnothing)\cup(Y_0^\delta\times Z_1)=\\&=((Y_2^\delta\cup Y_1^\delta)\times\varnothing)\cup(Y_0^\delta\times Z_1)=\alpha,\end{aligned}$$

if $Y_2^\delta\cup Y_1^\delta=Y_2^\alpha$ and $Y_0^\delta=Y_1^\alpha$ since $|Y_2^\delta|\ge 1$, $|Y_1^\delta|\ge 1$ and $|Y_0^\delta|\ge 1$.

$d')$ Let $|Y_2^\alpha|\ge 1$ then $|Y_1^\alpha|\ge 2$ $(|X|\ge 3)$. In this case we suppose, that

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$$\beta=\left(\left(\breve{D}\setminus Z_1\right)\times\varnothing\right)\cup\left(Z_1\times Z_1\right)\cup\left(\left(X\setminus\breve{D}\right)\times\breve{D}\right),$$

then $\beta\in B$ since $\left|X\setminus\breve{D}\right|\geq 1$ and

$$\begin{aligned}\delta\circ\beta&=\left(Y_2^\delta\times\varnothing\right)\cup\left(Y_1^\delta\times Z_1\beta\right)\cup\left(Y_0^\delta\times\breve{D}\beta\right)=\\&=\left(Y_2^\delta\times\varnothing\right)\cup\left(Y_1^\delta\times Z_1\right)\cup\left(Y_0^\delta\times Z_1\right)=\\&=\left(Y_2^\delta\times\varnothing\right)\cup\left(\left(Y_1^\delta\cup Y_0^\delta\right)\times Z_1\right)=\alpha,\end{aligned}$$

if $Y_2^\delta=Y_2^\alpha$ and $Y_1^\delta\cup Y_0^\delta=Y_1^\alpha$ since $\left|Y_2^\delta\right|\geq 1$, $\left|Y_1^\delta\right|\geq 1$ and $\left|Y_0^\delta\right|\geq 1$.

Therefore, the elements of the set B_3 are generating by elements of the set B.

The statement $a)$ of the Lemma 2.4 is proved.

3) Let $X=\breve{D}$, $\delta_0=\left(\left(X\setminus Z_1\right)\times Z_1\right)\cup\left(Z_1\times\breve{D}\right)$ and binary relation α be any element of the set B_3. Then $\delta_0\in B_1$ and quasinormal representation of a binary relation α has a form $\alpha=\left(Y_2^\alpha\times\varnothing\right)\cup\left(Y_1^\alpha\times Z_1\right)$, where $Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$. Now, let $\alpha=\delta\circ\beta$ for some $\delta,\beta\in\left(B\cup B_1\cup\{\gamma_0\}\right)\setminus\{\alpha\}$.

For the sets Y_2^α and Y_1^α we consider the following cases.

$e')$ let $Y_1^\alpha\geq 1$, then $Y_2^\alpha\geq 2$ ($|X|\geq 3$, by preposition) and $\delta\in B\setminus\{\alpha\}$, then by definition of a set B the quasinormal representation of a binary relation δ has a form $\delta=\left(Y_2^\delta\times\varnothing\right)\cup\left(Y_1^\delta\times Z_1\right)\cup\left(Y_0^\delta\times\breve{D}\right)$, where $Y_2^\delta,Y_1^\delta,Y_0^\delta\notin\{\varnothing\}$ since $V\left(X^*,\delta\right)=D$ and

$$\begin{aligned}\delta\circ\gamma_0&=\left(Y_2^\delta\times\varnothing\right)\cup\left(Y_1^\delta\times Z_1\gamma_0\right)\cup\left(Y_0^\delta\times\breve{D}\gamma_0\right)=\\&=\left(Y_2^\delta\times\varnothing\right)\cup\left(Y_1^\delta\times\varnothing\right)\cup\left(Y_0^\delta\times Z_1\right)=\left(\left(Y_2^\delta\cup Y_1^\delta\right)\times\varnothing\right)\cup\left(Y_0^\delta\times Z_1\right)=\alpha,\end{aligned}$$

if $Y_2^\delta\cup Y_1^\delta=Y_2^\alpha$ and $Y_0^\delta=Y_1^\alpha$ since $Y_2^\delta,Y_1^\delta,Y_0^\delta\notin\{\varnothing\}$.

$f')$ If $Y_2^\alpha\geq 1$, then $Y_1^\alpha\geq 2$ and

$$\begin{aligned}\delta_0\circ\gamma_0&=\left(\left(X\setminus Z_1\right)\times Z_1\gamma_0\right)\cup\left(Z_1\times\breve{D}\gamma_0\right)=\\&=\left(\left(X\setminus Z_1\right)\times\varnothing\right)\cup\left(Z_1\times Z_1\right)=\gamma_1,\\\delta\circ\gamma_1&=\left(Y_2^\delta\times\varnothing\right)\cup\left(Y_1^\delta\times Z_1\gamma_1\right)\cup\left(Y_0^\delta\times\breve{D}\gamma_1\right)=\\&=\left(Y_2^\delta\times\varnothing\right)\cup\left(Y_1^\delta\times Z_1\right)\cup\left(Y_0^\delta\times Z_1\right)=\left(Y_2^\delta\times\varnothing\right)\cup\left(\left(Y_1^\delta\cup Y_0^\delta\right)\times Z_1\right)=\alpha,\end{aligned}$$

if $Y_2^\delta=Y_2^\alpha$ and $Y_1^\delta\cup Y_0^\delta=Y_1^\alpha$ since $Y_2^\delta,Y_1^\delta,Y_0^\delta\notin\{\varnothing\}$.

Thus, if $X=\breve{D}$, then the elements of the set B_3 are generated by elements of the set $B\cup B_1\cup\{\gamma_0\}$.

The statement $b)$ of the Lemma 2.4 is proved.

4) Let $X=\breve{D}$ and α be arbitrary element of the set B_2. Then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_2^\alpha\times\varnothing\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Y_2^\alpha,Y_0^\alpha\notin\{\varnothing\}$. Now, let $\alpha=\delta\circ\beta$ for some $\delta,\beta\in\left(B_1\cup\{\gamma_0\}\right)\setminus\{\alpha\}$.

It is easy to see that the subsets Y_2^α and Y_0^α of the set X are two elements partitioning of the set X. Of this follows that $\delta=\left(Y_2^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$ is any element of the set B_1 and $\gamma_0\circ\delta_0\in B_2$ by definition of the binary relations γ_0 and δ_0.

$$\begin{aligned}\gamma_0\circ\delta_0&=\left(Z_1\times\varnothing\delta\right)\cup\left(\left(X\setminus Z_1\right)\times Z_1\delta\right)=\\&=\left(Z_1\times\varnothing\right)\cup\left(\left(X\setminus Z_1\right)\times\breve{D}\right)=\gamma_2,\\\delta\circ\gamma_2&=\left(Y_2^\alpha\times Z_1\gamma_2\right)\cup\left(Y_0^\alpha\times\breve{D}\gamma_2\right)=\left(Y_2^\alpha\times\varnothing\right)\cup\left(Y_0^\alpha\times\breve{D}\right)=\alpha\end{aligned}$$

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Of this follows that the elements of the set B_2 are generating by elements of the set $B_1 \cup \{\gamma_0\}$.

The statement $c)$ of the Lemma 2.4 is proved.

Lemma 2.4 is proved.

Lemma 2.5. *Let* $|X| \geq 3$, $D = \{\varnothing, Z_1, \breve{D}\} \in \Sigma_1(X,3)$,

$$B = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\} \text{ and } B_1 = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_1, \breve{D}\}\}.$$

If $|X \setminus \breve{D}| \geq 1$, *then the set* $B_1' = B \cup B_1$ *is irreducible generating set for the semi group* $B_X(D)$.

Proof. Let $|X| \geq 3$ and $|X \setminus \breve{D}| \geq 1$. First, we proved that every element of the semi group $B_X(D)$ is generating by elements of the set B_1'. Indeed, let α be arbitrary element of the semi group $B_X(D)$. Then quasinormal representation of a binary relation α has a form

$$\alpha = (Y_2^\alpha \times \varnothing) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}).$$

For the sets Y_2^α, Y_1^α and Y_0^α we consider the following cases:

1) $Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$. Then we have $V(X^*, \alpha) = D$, i.e. $\alpha \in B$.

2) $Y_2^\alpha = \varnothing$ $Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$. Then quasinormal representation of a binary relation α has a form $\alpha = (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, i.e. $\alpha \in B_1$.

3) $Y_1^\alpha = \varnothing$ $Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$. Then quasinormal representation of a binary relation α has a form $\alpha = (Y_2^\alpha \times \varnothing) \cup (Y_0^\alpha \times \breve{D})$. So, $\alpha \in B_2$. From the statement $a)$ of the Lemma 2.4 follows that the elements of the set B_2 are generating by elements of the set B.

4) $Y_0^\alpha = \varnothing$ $Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$. Then quasinormal representation of a binary relation α has a form $\alpha = (Y_2^\alpha \times \varnothing) \cup (Y_1^\alpha \times Z_1)$. So, $\alpha \in B_3$. From the statement $a)$ of the Lemma 2.4 follows, that the elements of the set B_3 is generating by elements of the set B.

5) If $Y_2^\alpha = Y_0^\alpha = \varnothing$, $Y_1^\alpha \neq \varnothing$, or $Y_2^\alpha = Y_1^\alpha = \varnothing$, $Y_0^\alpha \neq \varnothing$. Then quasinormal representation of a binary relation α has a form $\alpha = X \times Z_1$, or $\alpha = X \times \breve{D}$.

Let quasinormal representations of a binary relations δ and β_0 has a form $\delta = (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})$ and $\beta_0 = (\breve{D} \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D})$ where $Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$, i.e. $\delta, \beta_0 \in B_1$ and $Y_1^\delta \cup Y_0^\delta = X$ since $X \setminus \breve{D} \neq \varnothing$ (by assumption we have $|X \setminus \breve{D}| \geq 1$). So, the following equalities are true:

$$\begin{aligned} \delta \circ \beta_0 &= ((Y_1^\delta \times Z_1\beta_0) \cup (Y_0^\delta \times \breve{D}\beta_0)) = \\ &= (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times Z_1) = X \times Z_1 = \alpha. \end{aligned}$$

Now, let $\beta_1 = ((X \setminus Z_1) \times Z_1) \cup (Z_1 \times \breve{D})$, then $\beta_1 \in B_1$ since $Z_1 \neq \varnothing$ and $X \setminus Z_1 \neq \varnothing$ by definition of the semi lattice D. So, the following equalities are hold:

$$\begin{aligned} \delta \circ \beta_1 &= ((Y_1^\delta \times Z_1\beta_1) \cup (Y_0^\delta \times \breve{D}\beta_1)) = \\ &= (Y_1^\delta \times \breve{D}) \cup (Y_0^\delta \times \breve{D}) = X \times \breve{D} = \alpha. \end{aligned}$$

Therefore, the elements $\alpha = X \times Z_1$ and $\alpha = X \times \breve{D}$ are generating by elements of the set B_1.

6) $Y_1^\alpha = Y_0^\alpha = \varnothing$, then $Y_2^\alpha = X$ since the representation of a binary relation α is quasinirmal. Of this we have, that $\alpha = \varnothing$.

Now let $\delta = (Y_2^\delta \times \varnothing) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})$ is any element of the set B (by preposition the inequality $|X| \geq 3$ is true). Further, by assumption we have, that $|X \setminus \breve{D}| \geq 1$. In this case, for the $\beta_2 = (\breve{D} \times \varnothing) \cup ((X \setminus \breve{D}) \times \breve{D})$ we have $\beta_2 \in B_2$ and by statement $a)$ of the lemma 2.4 binary relation β_2 is generating by elements of the set B and

$$\delta \circ \beta_2 = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\beta_2\right) \cup \left(Y_0^\delta \times \breve{D}\beta_2\right) = \\ = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times \varnothing\right) \cup \left(Y_0^\delta \times \varnothing\right) = X \times \varnothing = \varnothing.$$

So, B_1' is generating set for the semi group $B_X(D)$.

By preposition $\left|X \setminus \breve{D}\right| \geq 1$ and we proved that the set B_1' is irreducible.

Let $\alpha \in B_1'$ and for the element α consider the following cases:

7) If $\alpha \in B$, then $\alpha \neq \sigma \circ \tau$ for all $\sigma, \tau \in B_X(D) \setminus \{\alpha\}$ since by statement $d)$ of the Lemma 2.1 follows that B is a set external elements for the semi group $B_X(D)$. So, $\alpha \neq \sigma \circ \tau$ for all $\sigma, \tau \in B_1' \setminus \{\alpha\}$ since $B_1' \setminus \{\alpha\} \subseteq B_X(D) \setminus \{\alpha\}$.

Thus, we have that $\alpha \notin B$.

8) If $\alpha \in B_1$, then by definition of a set B_1 the quasinormal representation of a binary relation α has a form $\alpha = \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$, where $Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$. Further, let $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B_1' \setminus \{\alpha\}$ and for the element δ consider the following cases:

$a')$ $\delta \in B \setminus \{\alpha\}$ and $\beta \in B_1' \setminus \{\alpha\}$. Then by definition of a set B the quasinormal representation of a binary relation δ has a form $\delta = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)$, where $Y_2^\delta, Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$ since $V\left(X^*, \delta\right) = D$ and

$$\left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right) = \alpha = \delta \circ \beta = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right).$$

But last equality is impossible since $Y_2^\alpha \notin \{\varnothing\}$.

So, we have that $\delta \notin B \setminus \{\alpha\}$.

$b')$ If $\delta \in B_1 \setminus \{\alpha\}$ and $\beta \in B_1' \setminus \{\alpha\}$. Then by definition of a set B_1 the quasinormal representation of a binary relation δ has a form $\delta = \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)$, where $Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$ and

$$\left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right) = \alpha = \delta \circ \beta = \left(Y_1^\delta \times Z_1\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right).$$

Last equalities are possible only if $Z_1\beta = Z_1$, $\breve{D}\beta = \breve{D}$ since $Z_1 \subset \breve{D}$. Of this we obtain, that

$$\alpha = \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right) = \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right) = \delta.$$

But the equality $\alpha = \delta$ contradict the condition $\delta \in B_1 \setminus \{\alpha\}$.

Thus, we have that $\delta \notin B_1 \setminus \{\alpha\}$.

So, from the cases $a')$ and $b')$ follows that $\alpha \notin B_1$.

Therefore, $\alpha \neq \delta \circ \beta$ for any $\delta, \beta \in B_1' \setminus \{\alpha\}$, i.e. the set $B_1' = B \cup B_1$ is irreducible generating set for the semigroup $B_X(D)$.

Lemma 2.5 is proved.

Lemma 2.6. *Let* $|X| \geq 3$, $D = \left\{\varnothing, Z_1, \breve{D}\right\} \in \Sigma_1(X, 3)$ *and*

$$B = \left\{\alpha \in B_X(D) \mid V\left(X^*, \alpha\right) = D\right\},\ B_1 = \left\{\alpha \in B_X(D) \mid V\left(X^*, \alpha\right) = \left\{Z_1, \breve{D}\right\}\right\}, \\ \gamma_0 = \left(Z_1 \times \varnothing\right) \cup \left(\left(X \setminus Z_1\right) \times Z_1\right).$$

If $X = \breve{D}$, *then the set* $B_2' = B \cup B_1 \cup \{\gamma_0\}$ *is irreducible generating set for the semigroup* $B_X(D)$.

Proof. Let $|X| \geq 3$, $X = \breve{D}$. First we proved that every element of the semigroup $B_X(D)$ is generating by elements of the set $B_2' = B \cup B_1 \cup \{\gamma_0\}$. Indeed, let α be any element of the semigroup $B_X(D)$. Then quasinormal representation of a binary relation α has a form

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$$\alpha=\left(Y_2^\alpha\times\varnothing\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right).$$

For a sets Y_2^α, Y_1^α and Y_0^α we consider the following cases.

1) $Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$. Then we have $V\left(X^*,\alpha\right)=D$, i.e. $\alpha\in B$;

2) $Y_2^\alpha=\varnothing$ $Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$. Then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, i.e. $\alpha\in B_1$;

3) $Y_1^\alpha=\varnothing$ $Y_2^\alpha,Y_0^\alpha\notin\{\varnothing\}$. Then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_2^\alpha\times\varnothing\right)\cup\left(Y_0^\alpha\times\breve{D}\right)\in B_2$ and by statement $c)$ of the Lemma 2.4 we have that the elements of the set B_2 are generating of elements of the set $B_1\cup\{\gamma_0\}$.

4) $Y_0^\alpha=\varnothing$ $Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$. Then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_2^\alpha\times\varnothing\right)\cup\left(Y_1^\alpha\times Z_1\right)\in B_3$. Then by statement $b)$ of the lemma 2.4 follows that elements of the set B_3 are generating by elements of the set $B\cup B_1\cup\{\gamma_0\}$;

5) If $Y_2^\alpha=Y_0^\alpha=\varnothing$, $Y_1^\alpha\neq\varnothing$ or $Y_2^\alpha=Y_1^\alpha=\varnothing$, $Y_0^\alpha\neq\varnothing$. Then quasinormal representtation of a binary relation α has a form $\alpha=X\times Z_1$, or $\alpha=X\times\breve{D}$.

If $\delta_0=\left(\left(X\setminus Z_1\right)\times Z_1\right)\cup\left(Z_1\times\breve{D}\right)$ and $\delta_1=\left(Z_1\times Z_1\right)\cup\left(\left(X\setminus Z_1\right)\times\breve{D}\right)$, then $\delta_0,\delta_1\in B_1$ since $|Z_1|\geq 1$ and $|X\setminus Z_1|\geq 1$ by definition of the semilattice D $\left(\varnothing\subset Z_1\subset\breve{D}\right)$ and

$$\begin{aligned}
\delta_0\circ\gamma_0&=\left(\left(X\setminus Z_1\right)\times\varnothing\right)\cup\left(Z_1\times Z_1\right)=\gamma_1\\
\delta_1\circ\gamma_1&=\left(Z_1\times Z_1\gamma_1\right)\cup\left(\left(X\setminus Z_1\right)\times\breve{D}\gamma_1\right)=\\
&=\left(Z_1\times Z_1\right)\cup\left(\left(X\setminus Z_1\right)\times Z_1\right)=X\times Z_1=\alpha,\\
\delta_0\circ\delta_0&=\left(\left(X\setminus Z_1\right)\times Z_1\delta_0\right)\cup\left(Z_1\times\breve{D}\delta_0\right)=\\
&=\left(\left(X\setminus Z_1\right)\times\breve{D}\right)\cup\left(Z_1\times\breve{D}\right)=X\times\breve{D}=\alpha.
\end{aligned}$$

So, the elements $\alpha=X\times Z_1$ and $\alpha=X\times\breve{D}$ are generating by elements of the set $B_1\cup\{\gamma_0\}$.

6) $Y_1^\alpha=Y_0^\alpha=\varnothing$. Then $Y_2^\alpha=X$ since the representation of the binary relation α is quasinormal. Then $\alpha=\varnothing$ and

$$\begin{aligned}
\gamma_0\circ\gamma_0&=\left(\left(Z_1\times\varnothing\right)\cup\left(\left(X\setminus Z_1\right)\times Z_1\right)\right)\circ\gamma_0=\left(Z_1\times\varnothing\gamma_0\right)\cup\left(\left(X\setminus Z_1\right)\times Z_1\gamma_0\right)=\\
&=\left(Z_1\times\varnothing\right)\cup\left(\left(X\setminus Z_1\right)\times\varnothing\right)=X\times\varnothing=\varnothing.
\end{aligned}$$

Thus, we have that the binary relation $\alpha=\varnothing$ is generating by elements of the set B_2'.

So, $B_2'=B\cup B_1\cup\{\gamma_0\}$ is generating set for the semigroup $B_X(D)$.

Now, let $|X|\geq 3$, $X=\breve{D}$ and we proved that the set $B_2'=B\cup B_1\cup\{\gamma_1\}$ is irreducible. For the element $\alpha\in B_2'$ consider the following cases.

7) If $\alpha\in B$, then $\alpha\neq\sigma\circ\tau$ for all $\sigma,\tau\in B_X(D)\setminus\{\alpha\}$ since by statement $d)$ of the Lemma 2.1 follows that B is a set external elements for the semigroup $B_X(D)$. So, $\alpha\neq\sigma\circ\tau$ for all $\sigma,\tau\in B_2'\setminus\{\alpha\}$ since $B_2'\setminus\{\alpha\}\subseteq B_X(D)\setminus\{\alpha\}$.

Thus we have $\alpha\notin B$.

8) Let $\alpha\in B_1$, then by definition of a set B_1 the quasinormal representation of a binary relation α has a form $\alpha=\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$. Further, let $\alpha=\delta\circ\beta$ for some $\delta,\beta\in B_2'\setminus\{\alpha\}$.

For the element δ consider the following cases:

$a')$ If $\delta\in B\setminus\{\alpha\}$ and $\beta\in B_2'\setminus\{\alpha\}$. Then by definition of a set B the quasinormal representation of a binary relation δ has a form $\delta=\left(Y_2^\delta\times\varnothing\right)\cup\left(Y_1^\delta\times Z_1\right)\cup\left(Y_0^\delta\times\breve{D}\right)$, where $Y_2^\delta,Y_1^\delta,Y_0^\delta\notin\{\varnothing\}$ since $V\left(X^*,\delta\right)=D$ and

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$$\left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right) = \alpha = \delta \circ \beta = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right).$$

But last equality is impossible since $Y_2^\delta \notin \{\varnothing\}$.

So, we have that $\delta \notin B \setminus \{\alpha\}$.

$b')$ If $\delta \in B_1 \setminus \{\alpha\}$ and $\beta \in B_2' \setminus \{\alpha\}$. Then by definition of a set B_1 the quasinormal representation of a binary relation δ has a form $\delta = \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)$, where $Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$ and

$$\left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right) = \alpha = \delta \circ \beta = \left(Y_1^\delta \times Z_1\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right).$$

Last equality is possible only if $Z_1 = Z_1\beta$ and $\breve{D} = \breve{D}\beta$ since $Z_1 \subset \breve{D}$, i.e.

$$\alpha = \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right) = \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right) = \delta.$$

We have $\alpha = \delta$, which contradict the condition $\delta \in B_1 \setminus \{\alpha\}$.

Thus, we have that $\delta \notin B_1 \setminus \{\alpha\}$.

$c')$ If $\delta = \gamma_0$ and $\beta \in B_2' \setminus \{\alpha\}$, then $\delta = (Z_1 \times \varnothing) \cup \left((X \setminus Z_1) \times Z_1\right)$ and $\delta \neq \alpha$, i.e.

$$\left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right) = \alpha = \delta \circ \beta = (Z_1 \times \varnothing) \cup \left((X \setminus Z_1) \times Z_1\beta\right).$$

But last equalities is impossible since $Y_1^\alpha, Z_1 \notin \{\varnothing\}$.

Thus, we have that $\delta \neq \gamma_0$.

Of the cases $a')$, $b')$ and $c')$ follows that $\alpha \notin B_1$.

9) $\alpha = \gamma_0 = (Z_1 \times \varnothing) \cup \left((X \setminus Z_1) \times Z_1\right)$. Further, let $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B_2' \setminus \{\gamma_0\}$.

For the element δ consider the following cases:

$a')$ If $\delta \in B \setminus \{\gamma_0\}$ and $\beta \in B_2' \setminus \{\gamma_0\}$. Then by definition of a set B the quasinormal representation of a binary relation δ has a form $\delta = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)$, where $Y_2^\delta, Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$ since $V\left(X^*, \delta\right) = D$ and

$$(Z_1 \times \varnothing) \cup \left((X \setminus Z_1) \times Z_1\right) = \alpha = \delta \circ \beta = \left(Y_2^\delta \times \varnothing\right) \cup \left(Y_1^\delta \times Z_1\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right).$$

Last equalities is possible only if $Z_1\beta = \varnothing$, $Z_1 = \breve{D}\beta$ or $Z_1 = Z_1\beta = \breve{D}\beta$.

If $Z_1\beta = \varnothing$, $Z_1 = \breve{D}\beta$, then by statement $a)$ of the Lemma 2.2 follows that $\left|X \setminus \breve{D}\right| \geq 1$. But, the inequality $\left|X \setminus \breve{D}\right| \geq 1$ contradict the equality $X = \breve{D}$.

If $Z_1 = Z_1\beta = \breve{D}\beta$, then by statement $b)$ of the Lemma 2.2 follows that $\left|X \setminus \breve{D}\right| \geq 1$. But, the inequality $\left|X \setminus \breve{D}\right| \geq 1$ contradict the equality $X = \breve{D}$.

Thus, in case $a')$ we have that $\delta \notin B \setminus \{\gamma_0\}$.

$b')$ If $\delta \in B_1 \setminus \{\gamma_0\}$ and $\beta \in B_2' \setminus \{\gamma_0\}$. Then by definition of a set B_1 the quasinormal representation of a binary relation δ has a form $\delta = \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)$, where $Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$ and

$$(Z_1 \times \varnothing) \cup \left((X \setminus Z_1) \times Z_1\right) = \alpha = \delta \circ \beta = \left(Y_1^\delta \times Z_1\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right).$$

Last equality is possible only if $Z_1\beta = \varnothing$ and $\breve{D}\beta = Z_1$ since $Z_1 \subset \breve{D}$.

If $Z_1\beta = \varnothing$ and $\breve{D}\beta = Z_1$ for some $\beta \in B$, then by statement $a)$ of the Lemma 2.2 we have $\left|X \setminus \breve{D}\right| \geq 1$. But last inequality contradict the condition $X = \breve{D}$.

Thus we have that $\delta \notin B_1 \setminus \{\gamma_0\}$.

Of the cases $a')$, $b')$ follows that $\alpha \neq \gamma_0$.

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Therefore, $\alpha \neq \delta \circ \beta$ for any $\delta, \beta \in B_2' \setminus \{\alpha\}$, i.e. the set $B_2' = B \cup B_1 \cup \{\gamma_0\}$ is irreducible generating set for the semigroup $B_X(D)$.

Lemma 2.6 is proved.

Lemma 2.7. *Let $|X| = 2$, $D = \{\varnothing, Z_1, \breve{D}\} \in \Sigma_1(X,3)$. Then $B = \varnothing$ and the set $B_3' = B_1 \cup \{\gamma_0\}$ is irreducible generating set for the semigroup $B_X(D)$.*

Proof. Let $X = \breve{D}$ and $|X| = 2$. Then $B_X(D) = \{\gamma_0, \alpha_1, \alpha_2, ..., \alpha_8\}$, where

$$\begin{aligned}
&\gamma_0 = (Z_1 \times \varnothing) \cup ((X \setminus Z_1) \times Z_1) = (X \setminus Z_1) \times Z_1,\\
&\alpha_1 = (Z_1 \times Z_1) \cup ((X \setminus Z_1) \times \breve{D}),\ \alpha_2 = ((X \setminus Z_1) \times Z_1) \cup (Z_1 \times \breve{D}),\\
&\alpha_3 = X \times \varnothing = \varnothing,\ \alpha_4 = ((X \setminus Z_1) \times \varnothing) \cup (Z_1 \times Z_1) = Z_1 \times Z_1,\\
&\alpha_5 = (Z_1 \times \varnothing) \cup ((X \setminus Z_1) \times \breve{D}) = (X \setminus Z_1) \times \breve{D},\\
&\alpha_6 = ((X \setminus Z_1) \times \varnothing) \cup (Z_1 \times \breve{D}) = Z_1 \times \breve{D},\ \alpha_7 = X \times Z_1,\ \alpha_8 = X \times \breve{D}.
\end{aligned}$$

In this case we have: $B = \varnothing$, $X = \breve{D}$, $B_1 = \{\alpha_1, \alpha_2\}$ and $B_3' = B_1 \cup \{\gamma_0\}$ is generating set for the semigroup $B_X(D)$. Indeed:

$\circ$	γ_0	α_1	α_2
γ_0	α_3	γ_0	α_5
α_1	γ_0	α_1	α_8
α_2	α_4	α_2	α_8

where $\alpha_2 \circ (\gamma_0 \circ \alpha_2) = \alpha_2 \circ \alpha_5 = \alpha_6$ and $(\alpha_1 \circ \alpha_2) \circ \gamma_0 = \alpha_8 \circ \gamma_0 = \alpha_7$. Of the last conditions and from the Lemma 2.6 we obtain that B_3' is irreducible generating set for the semigroup $B_X(D)$.

Lemma 2.7 is proved.

Theorem 2.2. *Let $|X| \geq 3$, $D = \{\varnothing, Z_1, \breve{D}\} \in \Sigma_1(X,3)$. If*

$$B = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\},\ B_1 = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_1, \breve{D}\}\},$$
$$\gamma_0 = (Z_1 \times \varnothing) \cup ((X \setminus Z_1) \times Z_1).$$

Then the following statements are true:

a) *if $|X \setminus \breve{D}| \geq 1$. Then the set $B \cup B_1$ is irreducible generating set for the semigroup $B_X(D)$;*

b) *if $X = \breve{D}$, then the set $B \cup B_1 \cup \{\gamma_0\}$ is irreducible generating set for the semigroup $B_X(D)$.*

c) *if $|X| = 2$, then the set $B_1 \cup \{\gamma_0\}$ is irreducible generating set for the semigroup $B_X(D)$.*

Proof. The statements $a)$, $b)$ and $c)$ immediately follows from the Lemma 2.5, 2.6 and 2.7 respectively.

Theorem 2.3. *Let $D = \{\varnothing, Z_1, \breve{D}\} \in \Sigma_1(X,3)$. If X is finite a set and $|X| = n$, then the following statements are true:*

a) *if $|X \setminus \breve{D}| \geq 1$, then the number $|B \cup B_1|$ of a set $B \cup B_1$ is equal to*

$$|B \cup B_1| = 3^n - 2^{n+1} + 1;$$

b) *if $|X| \geq 3$, $X = \breve{D}$, $\gamma_0 = (Z_1 \times \varnothing) \cup ((X \setminus Z_1) \times Z_1)$, then the number $|B \cup B_1 \cup \{\gamma_0\}|$ of a set $B \cup B_1 \cup \{\gamma_0\}$ is equal to*

$$|B \cup B_1 \cup \{\gamma_1\}| = 3^n - 2^{n+1} + 2;$$

c) *if $|X| = 2$, then then the number $|B_1 \cup \{\gamma_1, \gamma_2\}|$ of a set $B_1 \cup \{\gamma_0\}$ is equal to*

$$|B_1 \cup \{\gamma_0\}| = 3.$$

Proof. Let $B = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\}$ and

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$$\varphi_1=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\ \varphi_2=\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix},\ \varphi_3=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},$$
$$\varphi_4=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\ \varphi_5=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix},\ \varphi_6=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}.$$

If $\alpha\in B$, then quasinormal representation of a binary relation α has a form $\alpha=\left(Y^{\alpha}_{\varphi_j(1)}\times\varnothing\right)\cup\left(Y^{\alpha}_{\varphi_j(2)}\times Z_1\right)\cup\left(Y^{\alpha}_{\varphi_j(3)}\times\breve{D}\right)$, where $j=1,2,...,5,6$ and a system of subsets $Y^{\alpha}_{\varphi_j(1)},Y^{\alpha}_{\varphi_j(2)},Y^{\alpha}_{\varphi_j(3)}\notin\{\varnothing\}$ of the set X is partitioning of the set X. Then the number k_n^3 partitioning $Y^{\alpha}_{\varphi_j(1)},Y^{\alpha}_{\varphi_j(2)},Y^{\alpha}_{\varphi_j(3)}$ of the set X for fixed j $(1\le j\le 6)$ is equal to

$$k_n^3=\sum_{i=1}^{3}\frac{(-1)^{3+i}}{(i-1)!(3-i)!}\cdot i^{n-1}=\frac{1}{2}\cdot 3^{n-1}-2^{n-1}+\frac{1}{2}.$$

(see [1], Theorem 1.17.1). Of this obtain that $|B|=6\cdot k_n^3=3^n-3\cdot 2^n+3$.

If $\alpha\in B_1$, then quasinormal representation of a binary relation α has a form $\alpha=\left(Y_1^{\alpha}\times Z_1\right)\cup\left(Y_0^{\alpha}\times\breve{D}\right)$, where a system $Y_1^{\alpha},Y_0^{\alpha}$ is partitioning of the set X. By definition of a set B_1 we obtain $B_1=B_X(D')\setminus\{X\times Z_1,X\times\breve{D}\}$, where $D'=\{Z_1,\breve{D}\}$. So, we have, $|B_1|=|B_X(D')|-2=2^{|X|}-2=2^n-2$. By definition of a sets B, B_1 and $\{\gamma_0\}$ follows that $B\cap B_1=B\cap\{\gamma_0\}=B_1\cap\{\gamma_0\}=\varnothing$. Of this we obtain that:

$$|B\cup B_1|=\left(3^n-3\cdot 2^n+3\right)+\left(2^n-2\right)=3^n-2^{n+1}+1,$$

if $|X\setminus\breve{D}|\ge 1$;

$$|B\cup B_1\cup\{\gamma_0\}|=\left(3^n-2^{n+1}+1\right)+1=3^n-2^{n+1}+2,$$

if $|X|\ge 3$, $X=\breve{D}$;

$$|B_1\cup\{\gamma_0\}|=2^{|2|}-2+1=3,$$

if $|X|=2$.

Theorem 2.3 is proved.

Example 2.1. Let $X=\{1,2,3\}$, $Z_1=\{1\}$, $\breve{D}=\{1,2\}$, $D=\{\varnothing,Z_1,\breve{D}\}$ and $|X\setminus\breve{D}|=1$. Then $B_X(D)=\{\alpha_1,\alpha_2,...,\alpha_{27}\}$, where

$$\alpha_1=(Z_1\times\varnothing)\cup\left((\breve{D}\setminus Z_1)\times Z_1\right)\cup\left((X\setminus\breve{D})\times\breve{D}\right),$$
$$\alpha_2=(Z_1\times\varnothing)\cup\left((X\setminus\breve{D})\times Z_1\right)\cup\left((\breve{D}\setminus Z_1)\times\breve{D}\right),$$
$$\alpha_3=\left((\breve{D}\setminus Z_1)\times\varnothing\right)\cup(Z_1\times Z_1)\cup\left((X\setminus\breve{D})\times\breve{D}\right),$$
$$\alpha_4=\left((\breve{D}\setminus Z_1)\times\varnothing\right)\cup\left((X\setminus\breve{D})\times Z_1\right)\cup\left(Z_1\times\breve{D}\right),$$
$$\alpha_5=\left((X\setminus\breve{D})\times\varnothing\right)\cup\left((\breve{D}\setminus Z_1)\times Z_1\right)\cup\left(Z_1\times\breve{D}\right),$$
$$\alpha_6=\left((X\setminus\breve{D})\times\varnothing\right)\cup(Z_1\times Z_1)\cup\left((\breve{D}\setminus Z_1)\times\breve{D}\right),$$
$$\alpha_7=(Z_1\times Z_1)\cup\left(\{2,3\}\times\breve{D}\right),\ \alpha_8=\left((\breve{D}\setminus Z_1)\times Z_1\right)\cup\left(\{1,3\}\times\breve{D}\right),$$
$$\alpha_9=\left((X\setminus\breve{D})\times Z_1\right)\cup\left(\breve{D}\times\breve{D}\right),\ \alpha_{10}=\left(\breve{D}\times Z_1\right)\cup\left((X\setminus\breve{D})\times\breve{D}\right),$$
$$\alpha_{11}=(\{1,3\}\times Z_1)\cup\left((\breve{D}\setminus Z_1)\times\breve{D}\right),\ \alpha_{12}=(\{2,3\}\times Z_1)\cup\left(Z_1\times\breve{D}\right),$$
$$\alpha_{13}=\left(\breve{D}\times\varnothing\right)\cup\left((X\setminus\breve{D})\times Z_1\right),\ \alpha_{14}=\left((X\setminus\breve{D})\times\varnothing\right)\cup\left(\breve{D}\times Z_1\right),$$
$$\alpha_{15}=\left(\breve{D}\times\varnothing\right)\cup\left((X\setminus\breve{D})\times\breve{D}\right),\ \alpha_{16}=(\{1,3\}\times\varnothing)\cup\left((\breve{D}\setminus Z_1)\times\breve{D}\right),$$
$$\alpha_{17}=(\{2,3\}\times\varnothing)\cup\left(Z_1\times\breve{D}\right),\ \alpha_{18}=\left((\breve{D}\setminus Z_1)\times\varnothing\right)\cup\left(\{1,3\}\times\breve{D}\right),$$
$$\alpha_{19}=\left((X\setminus\breve{D})\times\varnothing\right)\cup\left(\breve{D}\times\breve{D}\right),\ \alpha_{20}=(Z_1\times\varnothing)\cup\left(\{2,3\}\times\breve{D}\right),$$
$$\alpha_{21}=(\{1,3\}\times\varnothing)\cup\left((\breve{D}\setminus Z_1)\times Z_1\right),\ \alpha_{22}=(Z_1\times\varnothing)\cup(\{2,3\}\times Z_1),$$
$$\alpha_{23}=(\{2,3\}\times\varnothing)\cup(Z_1\times Z_1),\ \alpha_{24}=\left((\breve{D}\setminus Z_1)\times\varnothing\right)\cup(\{1,3\}\times Z_1),$$
$$\alpha_{25}=\varnothing,\ \alpha_{26}=\{1,2,3\}\times Z_1,\ \alpha_{27}=\{1,2,3\}\times\breve{D}.$$

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$B=\{\alpha_1,\alpha_2,...,\alpha_6\}$, $B_1=\{\alpha_7,\alpha_8,...,\alpha_{12}\}$ and $|B\cup B_1|=12$.

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
α_1	α_{13}	α_{15}	α_{22}	α_{20}	α_{20}	α_1	α_1	α_{20}	α_{20}	α_{22}	α_1	α_{20}
α_2	α_{21}	α_{16}	α_{22}	α_{20}	α_{20}	α_2	α_2	α_{20}	α_{20}	α_{22}	α_2	α_{20}
α_3	α_{13}	α_{15}	α_{24}	α_{18}	α_{18}	α_{18}	α_3	α_{18}	α_{18}	α_{24}	α_3	α_{18}
α_4	α_{23}	α_{17}	α_{24}	α_{18}	α_{18}	α_4	α_4	α_{18}	α_{18}	α_{24}	α_4	α_{18}
α_5	α_{23}	α_{17}	α_{14}	α_{19}	α_{19}	α_5	α_5	α_{19}	α_{19}	α_{14}	α_5	α_{19}
α_6	α_{21}	α_{16}	α_{14}	α_{19}	α_{19}	α_6	α_6	α_{19}	α_{19}	α_{14}	α_6	α_{19}
α_7	α_{22}	α_{20}	α_{26}	α_{27}	α_{27}	α_7	α_7	α_{27}	α_{27}	α_{27}	α_7	α_{27}
α_8	α_{24}	α_{24}	α_{26}	α_{27}	α_{27}	α_8	α_8	α_{27}	α_{27}	α_{26}	α_8	α_{27}
α_9	α_{14}	α_{19}	α_{26}	α_9	α_{27}	α_9	α_9	α_{27}	α_{27}	α_{26}	α_9	α_{27}
α_{10}	α_{13}	α_{15}	α_{26}	α_{27}	α_{27}	α_{10}	α_{10}	α_{27}	α_{27}	α_{26}	α_{10}	α_{27}
α_{11}	α_{21}	α_{16}	α_{26}	α_{27}	α_{27}	α_{11}	α_{11}	α_{27}	α_{27}	α_{26}	α_{11}	α_{27}
α_{12}	α_{17}	α_{23}	α_{26}	α_{27}	α_{27}	α_{12}	α_{12}	α_{27}	α_{27}	α_{26}	α_{12}	α_{27}

In this case we have $\alpha_2\circ\alpha_3\circ\alpha_2=\alpha_{22}\circ\alpha_2=\alpha_{25}$, i.e. the set $B\cup B_1$ is irreducible generated set for the semi group $B_X(D)$.

Example 2.2. Let $X=\{1,2,3\}=\breve{D}$, $Z_1=\{1,2\}$, $D=\{\varnothing,Z_1,\breve{D}\}$ $X=\breve{D}$. Then $B_X(D)=\{\alpha_1,\alpha_2,...,\alpha_{27}\}$, where

$$\alpha_1=(\{1\}\times\varnothing)\cup(\{2\}\times Z_1)\cup(\{3\}\times\breve{D}),\ \alpha_2=(\{1\}\times\varnothing)\cup(\{3\}\times Z_1)\cup(\{2\}\times\breve{D}),$$
$$\alpha_3=(\{2\}\times\varnothing)\cup(\{1\}\times Z_1)\cup(\{3\}\times\breve{D}),\ \alpha_4=(\{2\}\times\varnothing)\cup(\{3\}\times Z_1)\cup(\{1\}\times\breve{D}),$$
$$\alpha_5=(\{3\}\times\varnothing)\cup(\{2\}\times Z_1)\cup(\{1\}\times\breve{D}),\ \alpha_6=(\{3\}\times\varnothing)\cup(\{1\}\times Z_1)\cup(\{2\}\times\breve{D}),$$
$$\alpha_7=(\{1\}\times Z_1)\cup(\{2,3\}\times\breve{D}),\ \alpha_8=(\{2\}\times Z_1)\cup(\{1,3\}\times\breve{D}),$$
$$\alpha_9=(\{3\}\times Z_1)\cup(Z_1\times\breve{D}),\ \alpha_{10}=(Z_1\times Z_1)\cup(\{3\}\times\breve{D}),$$
$$\alpha_{11}=(\{1,3\}\times Z_1)\cup(\{2\}\times\breve{D}),\ \alpha_{12}=(\{2,3\}\times Z_1)\cup(\{1\}\times\breve{D}),$$
$$\alpha_{13}=(\{3\}\times\varnothing)\cup(Z_1\times Z_1),\ \alpha_{14}=(Z_1\times\varnothing)\cup(\{3\}\times\breve{D}),$$
$$\alpha_{15}=(Z_1\times\varnothing)\cup(\{3\}\times Z_1)=\gamma_0,\ \alpha_{16}=(\{1,3\}\times\varnothing)\cup(\{2\}\times\breve{D}),$$
$$\alpha_{17}=(\{2,3\}\times\varnothing)\cup(\{1\}\times\breve{D}),\ \alpha_{18}=(\{2\}\times\varnothing)\cup(\{1,3\}\times\breve{D}),$$
$$\alpha_{19}=(\{3\}\times\varnothing)\cup(Z_1\times\breve{D}),\ \alpha_{20}=(\{1\}\times\varnothing)\cup(\{2,3\}\times\breve{D}),$$
$$\alpha_{21}=(\{1,3\}\times\varnothing)\cup(\{2\}\times Z_1),\ \alpha_{22}=(\{1\}\times\varnothing)\cup(\{2,3\}\times Z_1),$$
$$\alpha_{23}=(\{2,3\}\times\varnothing)\cup(\{1\}\times Z_1),\ \alpha_{24}=(\{2\}\times\varnothing)\cup(\{1,3\}\times Z_1),$$
$$\alpha_{25}=\varnothing,\ \alpha_{26}=\{1,2,3\}\times Z_1,\ \alpha_{27}=\{1,2,3\}\times\breve{D}.$$

$B=\{\alpha_1,\alpha_2,...,\alpha_6\}$, $B_1=\{\alpha_7,\alpha_8,...,\alpha_{12}\}$, $\gamma_0=\alpha_{15}$ and $|B\cup B_1\cup\{\gamma_0\}|=13$.

$\circ$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	γ_0
α_1	α_1	α_{20}	α_1	α_{20}	α_{20}	α_{20}	α_{20}	α_{20}	α_{20}	α_1	α_{20}	α_{20}	γ_0
α_2	α_2	α_{20}	α_2	α_{20}	α_{20}	α_{20}	α_{20}	α_{20}	α_{20}	α_{20}	α_{20}	α_{20}	α_{21}
α_3	α_3	α_{18}	α_3	α_4	α_{17}	α_{18}	α_{18}	α_{18}	α_{18}	α_3	α_4	α_4	γ_0
α_4	α_4	α_{18}	α_4	α_4	α_4	α_{18}	α_{18}	α_{18}	α_{18}	α_4	α_{18}	α_{18}	α_{23}
α_5	α_5	α_{19}	α_5	α_{17}	α_5	α_{19}	α_{19}	α_{19}	α_{19}	α_5	α_{19}	α_{19}	α_{23}
α_6	α_6	α_{19}	α_6	α_{19}	α_{19}	α_{19}	α_{19}	α_{19}	α_{19}	α_6	α_{19}	α_5	α_{21}
α_7	α_7	α_9	α_7	α_{27}	α_{27}	α_{27}	α_7	α_{27}	α_{27}	α_7	α_{27}	α_{27}	α_{22}

α_8	α_8	α_{27}	α_8	α_8	α_8	α_{27}	α_{27}	α_8	α_{27}	α_8	α_7	α_{27}	α_{22}
α_9	α_9	α_{27}	α_9	α_{27}	α_{27}	α_{27}	α_{27}	α_{27}	α_9	α_{26}	α_{27}	α_{27}	α_{13}
α_{10}	α_{10}	α_9	α_{10}	α_{27}	α_{27}	α_{27}	α_{27}	α_{27}	α_{27}	α_{10}	α_{27}	α_{27}	γ_0
α_{11}	α_{11}	α_{19}	α_{11}	α_{27}	α_{27}	α_{27}	α_{27}	α_{27}	α_{27}	α_{11}	α_{27}	α_{27}	α_{21}
α_{12}	α_{12}	α_{27}	α_{12}	α_{27}	α_{27}	α_{27}	α_{27}	α_{27}	α_{27}	α_{12}	α_{27}	α_{27}	α_{23}
γ_0	γ_0	α_{14}	γ_0	α_{14}	α_{14}	α_{14}	α_{14}	α_{14}	α_{14}	α_{14}	α_{14}	α_{14}	α_{25}

In this case we have: $\alpha_{11} \circ \alpha_{15} \circ \alpha_9 = \alpha_{11} \circ \alpha_{14} = \alpha_{16}$ and $\alpha_3 \circ \alpha_9 \circ \alpha_{15} = \alpha_{18} \circ \alpha_{15} = \alpha_{24}$, i.e. the set $B \cup B_1 \cup \{\gamma_0\}$ is irreducible generated set for the semi group $B_X(D)$.

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