GENERATED SETS OF THE COMPLETE SEMIGROUP BINARI RELATIONS DEFINED BY SEMILATTICES OF THE CLASS \( \Sigma_1(X,2) \)

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Abstract. In this article, we study generated sets of the complete semigroup \( B_X(D) \) defined by an \( X - \) semi lattice \( D \) of the class \( \Sigma_1(X,2) \).

Key words: Semi group, semi lattice, binary relation.

I. INTRODUCTION

1.1. Let \( X \) be an arbitrary nonempty set, \( D \) is an \( X - \) semilattice of unions which closed with respect to the set-theoretic union of elements from \( D \), \( f \) be an arbitrary mapping of the set \( X \) in the set \( D \). To each mapping \( f \) we put into correspondence a binary relation \( \alpha_f \) on the set \( X \) that satisfies the condition

\[
\{ x \in X \mid \forall \alpha \in \alpha_f \}
\]

The set of all such \( \alpha_f \) \(( f : X \rightarrow D )\) is denoted by \( B_X(D) \). It is easy to prove that \( B_X(D) \) is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an \( X - \) semilattice of unions \( D \).

We denote by \( \emptyset \) an empty binary relation or an empty subset of the set \( X \). The condition \(( x, y ) \in \alpha \) will be written in the form \( x \alpha y \). Further, let \( x, y \in X , Y \subseteq X , \alpha \in B_X(D), \ D = \bigcup_{y \in D} Y \) and \( T \in D \). We denote by the symbols \( y \alpha , Y \alpha , V(D, \alpha) , X^*, V\left( X^*, \alpha \right) \) and \( D_t \) the following sets:

\[
\begin{align*}
\{ x \in X \mid y \alpha x \}, & \quad Y \alpha = \bigcup_{y \in T} Y \alpha, \quad V(D, \alpha) = \{ Y \alpha \mid Y \in D \}, \\
X^* = \{ Y \mid \emptyset \neq Y \subseteq X \}, & \quad V\left( X^*, \alpha \right) = \{ Y \alpha \mid \emptyset \neq Y \subseteq X \}, \\
D_t = \{ Z \in D \mid T \subseteq Z \}. &
\end{align*}
\]

It is well know the following statements:

Theorem 1.2. Let \( D = \{ D, Z_1, Z_2, \ldots, Z_m \} \) be some finite \( X - \) semilattice of unions and \( C(D) = \{ P_0, P_1, P_2, \ldots, P_m \} \) be the family of sets of pairwise nonintersecting subsets of the set \( X \) (the set \( \emptyset \) can be repeat several time). If \( \varphi \) is a mapping of the semilattice \( D \) on the family of sets \( C(D) \) which satisfies the condition

\[
\varphi = \left( \begin{array}{cccc}
D & Z_1 & Z_2 & \ldots & Z_m \\
\hline
P_0 & P_1 & P_2 & \ldots & P_m
\end{array} \right)
\]

and \( \hat{D}_2 = D \setminus D_2 \), then the following equalities are valid:
In the sequel these equalities will be called formal. It is proved that if the elements of the semilattice $D$ are represented in the form (1.1), then among the parameters $P_i$ $(0 < i \leq m-1)$ there exist such parameters that cannot be empty sets for $D$. Such sets $P_i$ are called basis sources, whereas sets $P_j$ $(0 \leq j < m-1)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping $\varphi$ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping $\varphi$ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1], chapter 11). Let $P_0, P_1, P_2, ..., P_{m-1}$ be parameters in the formal equalities, $\beta \in B_x(D)$ and 

$$\tilde{\beta} = \bigcup_{i=0}^{m-1} P_i \times \bigcup_{i \in \beta} \bigcup_{t \in X \setminus D} \{t' \times t' \beta\}. \quad \ldots (1.2)$$

The representation of the binary relation $\beta$ of the form $\tilde{\beta}$ will be called subquasinormal. If $\tilde{\beta}$ be the subquasinormal representation of the binary relation $\beta$, then for the binary relation $\tilde{\beta}$ the following statements are true:

a) $\tilde{\beta} \in B_x(D);$  
b) $\beta \subseteq \tilde{\beta};$

c) the subquasinormal representation of the binary relation $\beta$ is quasinormal;

d) if $\tilde{\beta} = \left\{ P_0, P_1, P_2, ..., P_{m-1} \right\}$, then $\tilde{\beta}$ is a mapping of the family of sets $C(D)$ in the set $D \cup \{\emptyset\}.$

e) if $\tilde{\beta} : X \setminus D \rightarrow D$ is a mapping satisfying the condition $\tilde{\beta}(t') = t' \beta$ for all $t' \in X \setminus D$, then

$$\tilde{\beta} = \bigcup_{i=0}^{m-1} P_i \times \bigcup_{i \in \beta} \bigcup_{t \in X \setminus D} \{t' \times \tilde{\beta}(t')\}. \quad \ldots (1.3)$$

Remark, that if $P_j$ $(0 \leq j < m-1)$ is such completeness sources, that $P_0 = \emptyset$, then the equality $P_0 \tilde{\beta} = \emptyset$ always is hold. There also exists such a basic sources $P_i$ $(0 \leq i < m-1)$ for which $\bigcup_{t \in P_i} t \beta = \emptyset$, i.e. $P_0 \tilde{\beta} = \emptyset$.

**Example 1.1.** Let $X = \{1, 2, 3, 4\}$, $D = \{\emptyset, \{1, 2\}\}$, then $P_0 = \emptyset$, $P_2 = \{1, 2\}$. If $\tilde{\beta} = \{\{1\}, \{1, 2\}, \{2, 1\}, \{2, 2\}, \{4, 1\}, \{4, 2\}\}$, then $\beta \in B_x(D)$ and subquasinormal representation of a binary relation $\beta$ has a form

Fig. 1.1

$$\tilde{\beta} = \left\{ P_0, P_1 \right\}, \quad \tilde{\beta} = \left\{ P_1, 3 \right\}.$$  

$$\bar{\beta} = (\emptyset \times \emptyset) \cup (\{1, 2\} \times \{1, 2\}) \cup \{3\} \times \emptyset \cup \{4\} \times \{1, 2\} =$$

$$= (\emptyset \times \emptyset) \cup (P_1 \times \{1, 2\}) \cup \{3\} \times \emptyset \cup \{4\} \times \{1, 2\},$$

where $P_i$ are basic sources and $\beta$ is completeness sources.

**Theorem 1.2.** Let $\alpha, \beta \in B_x(D)$, then $\alpha \circ \beta = \alpha \circ \tilde{\beta}$ (see [4], Proposition 2).

2.1 Let $\Sigma_i(X, 2)$ be a class of all $X$ - semilattices of unions, whose every element is isomorphic to an $X$ - semilattice of unions $D = \{Z_i, \bar{D}\}$ which satisfies the condition $Z_i \subset \bar{D}$ (see, Fig. 2.1).
Let $C(D) = \{P_0, P_1\}$, where $P_0, P_1 \subseteq X$ and $P_0 \cap P_1 = \emptyset$ and $\varphi = \left( \begin{array}{c} \bar{D} \\ \bar{Z}_i \end{array} \right)$ is a mapping of the semi lattice $D$ onto the set $C(D)$. Then for the formal equalities of the semilattice $D$ we have a form:

$$
\bar{D} = P_0 \cup P_1,
\bar{Z}_i = P_0,
$$

(2.1)

Here the element $P_1$ be basis sources, the element $P_0$ are sources of completeness of the semilattice $D$. Therefore $|X| \geq 1$.

**Definition 2.1.** We say that an element $\alpha$ of the semigroup $B_\alpha(D)$ is external if $\alpha \neq \delta \circ \beta$ for all $\delta, \beta \in B_\alpha(D) \setminus \{\alpha\}$ (see [1], Definition 1.15.1).

It is well known, that if $B$ is all external elements of the semigroup $B_\alpha(D)$ and $B'$ be any generated set for the $B_\alpha(D)$, then $B \subseteq B'$ (see [1], Lemma 1.15.1).

**Lemma 2.1.** Let $D = \{Z_i, \bar{D}\} \in \Sigma, (X, 2)$, $B = \{\alpha \in B_\alpha(D)|V(X^*, \alpha) = D\}$. If $B \neq \emptyset$, then $B$ is a set external elements of the semigroup $B_\alpha(D)$.

**Proof.** Let $D = \{Z_i, \bar{D}\} \in \Sigma, (X, 2)$, $\alpha \in B$ and $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B_\alpha(D) \setminus \{\alpha\}$. Then element $\delta$ has quasinormal representation of a form $\delta = (Y_{i}^f \times Z_i) \cup (Y_{0}^f \times \bar{D})$, i.e. $Y_{i}^f \cup Y_{0}^f = X$ and $Y_i^f \cap Y_0^f = \emptyset$. (see [1], definition 1.11), By Theorem 1.2 follows that $\alpha = \delta \circ \beta = \delta \circ \bar{\beta}$, where $\bar{\beta}$ is subquasinormal representation of a binary relation $\beta$. It is easy to see, that

$$
\alpha = \delta \circ \bar{\beta} = (Y_{i}^f \times Z_i \bar{\beta}) \cup (Y_{0}^f \times \bar{D} \bar{\beta})
$$

From the equality $\alpha = \delta \circ \bar{\beta}$ follows that $D = V(X^*, \alpha) \subseteq V(D, \bar{\beta})$ (see [1], Theorem 4.1.1). So, $D = V(D, \bar{\beta})$.

By proposition $\alpha \in B$, i.e. there exists quasinormal representations of a binary relation $\alpha$ of the form $\alpha = (Y_{i}^f \times Z_i) \cup (Y_{0}^f \times \bar{D})$, where $|Y_{i}^f| \geq 1$ for all $i, 0, 1$ since $\alpha \in B$ (if $Y_j^f = \emptyset$ for some $j (0 \leq j \leq 1)$, then $V(X^*, \alpha) \neq D$), i.e. $|X| \geq 2$.

For the element $Z_i$ we consider the following cases.

a) $Z_i = \emptyset$. In this case we have $P_0 = \cap D = \emptyset$ and

$$
\gamma_1 = \left( \begin{array}{c} \emptyset \\ \emptyset \end{array} \right), \gamma_2 = \left( \begin{array}{c} \emptyset \\ \emptyset \end{array} \right)
$$

are all mappings of the set $\{\emptyset, P_1\}$ in the semi lattice $D$ satisfying condition $\gamma_i(\emptyset) = \emptyset (i = 1, 2)$.

If $X = \bar{D}$, then from the formal equalities (2.1) follows that $\emptyset \cup P_1 = \bar{D}$. In this case $P_0 = \emptyset$, $P_1 = \bar{D}$, $X \setminus \bar{D} = \emptyset$, $\bar{\beta} = (\emptyset \times \emptyset) \cup (P_1 \times \bar{D}) \cup \emptyset = X \times \bar{D}$ (see equality 1.2). It easy to see $V(D, \bar{\beta}) = D$ and

$$
\alpha = \delta \circ \bar{\beta} = (Y_{i}^f \times Z_i \circ (X \times D)) \cup (Y_{0}^f \times \bar{D} \circ (X \times D)) =
(Y_{i}^f \times Z_i \circ (X \times D)) \cup (Y_{0}^f \times \bar{D} \circ (X \times D)) =
(Y_{i}^f \times D) \cup (Y_{0}^f \times D) = X \times \bar{D} \notin B
$$

since $V(X^*, \alpha) = \{\bar{D}\} \neq D$. So, $X \neq \bar{D}$.

In the sequel we suppose, that $|X \setminus \bar{D}| \geq 1$.

For the binary relations $\gamma_i (i = 1, 2)$ we consider the following cases.
Let $\gamma_1 = \left( \emptyset, P_1, D \right)$. By formal equality (2.1) follows that $P_1 = D \neq \emptyset$. If $\gamma_i$ be a mapping of the set $X \setminus D$ on the set $D \setminus \emptyset = \{D\}$ (by preposition $|X \setminus D| \geq 1$), then

$$\bar{\beta} = (P_1 \times \emptyset) \cup \bigcup_{t \in X \setminus D} \left( \{t\} \times \gamma_i(t) \right). \quad \ldots (2.3)$$

$V(D, \bar{\beta}) = D$ (see equality 1.2) and from the formal equality (2.1) and equalities (2.2), (2.3) follows that

$$Z_i \bar{\beta} = \emptyset \bar{\beta} = \emptyset,$$

$$D \bar{\beta} = \emptyset \bar{\beta} \cup P_1 \bar{\beta} = \emptyset \cup \emptyset = \emptyset,$$

$$\alpha = \delta \circ \bar{\beta} = (Y_i^\delta \times Z_i \bar{\beta}) \cup (Y_i^\delta \times D \bar{\beta}) =$$

$$\quad = (Y_i^\delta \times \emptyset) \cup (Y_i^\delta \times \emptyset) = X \times \emptyset = \emptyset \not\in B$$

since $V(X^*, \alpha) = \{\emptyset\} \neq D$.

If $\gamma_2 = \left( \emptyset, P_2, \emptyset \right)$ and $\gamma_2$ be a mapping of the set $X \setminus D$ on the semilattice $D \setminus \{D\}$. So, if

$$\bar{\beta} = (P_1 \times \emptyset) \cup \bigcup_{t \in X \setminus D} \left( \{t\} \times \gamma_i(t) \right), \quad \ldots (2.4)$$

then $V(D, \bar{\beta}) = D$. From the formal equality (2.1) and equalities (2.2), (2.4) we have:

$$Z_i \bar{\beta} = \emptyset \bar{\beta} = \emptyset,$$

$$D \bar{\beta} = \emptyset \bar{\beta} \cup P_1 \bar{\beta} = \emptyset \cup \emptyset = \emptyset,$$

$$\alpha = \delta \circ \bar{\beta} = (Y_i^\delta \times Z_i \bar{\beta}) \cup (Y_i^\delta \times D \bar{\beta}) =$$

$$\quad = (Y_i^\delta \times \emptyset) \cup (Y_i^\delta \times \emptyset) = X \times \emptyset = \emptyset \not\in B.

But, the equality $\alpha = \delta$ contradict the condition, that $\delta \in B_\times(D) \setminus \{\alpha\}$. That is in this case $\alpha \not\in B$.

So, from the cases a) follows that $B$ is a set external elements of semigroup $B_\times(D)$ since the mappings $\gamma_i$ ($i=1,2$) are all mappings of the set $\{\emptyset, P_1\}$ in the semilattice $D$ satisfying condition $\gamma_i(\emptyset) = \emptyset$.

b) $Z_i \neq \emptyset$. Then $P_0 = \cap D \neq \emptyset$ and

$$\sigma_1 = \left( P_0, P_1, Z_i \right), \quad \sigma_2 = \left( P_0, P_1, D \right), \quad \sigma_3 = \left( P_0, P_1, Z_i \right) \quad \text{and} \quad \sigma_4 = \left( P_0, P_1, D \right)$$

are all mappings of the set $\{P_0, P_1\}$ in the semilattice $D$. .

If $X = \bar{D}$ and $\sigma_i(P_1) = \sigma_i(P_1)$ ($i = 1, 4$), then from the formal equalities (2.1) follows that $\sigma_i(P_1) = \bar{D}$ and in this case $\bar{\beta} = X \times \sigma_i(P_1)$ (see equality 1.2). So, $\bar{\beta} = X \times Z_i$ or $\bar{\beta} = X \times D$. Both case $V(D, \bar{\beta}) = D$ and

$$\alpha = \delta \circ \bar{\beta} = (Y_i^\delta \times Z_i) \cup \left( Y_i^\delta \times \bar{D} \right) \circ (X \times Z_i) =$$

$$\quad = (Y_i^\delta \times Z_i) \cup \left( Y_i^\delta \times \bar{D} \right) \circ (X \times Z_i) =$$

$$\quad = (Y_i^\delta \times Z_i) \cup \left( Y_i^\delta \times Z_i \right) = X \times \emptyset \not\in B.$$
Let $\sigma_1 = \begin{pmatrix} P_0 & P_1 \\ Z_0 & Z_1 \end{pmatrix}$. In this case we have $P_0 \neq \emptyset$ and $P_1 \neq \emptyset$. If $\sigma_1$ be a mapping of the set $X \setminus \bar{D}$ on the set $D \setminus \{Z_1\} = \{D\}$ (by preposition $|X \setminus \bar{D}| \geq 1$), then

$$\bar{\beta} = ((P_0 \cup R) \times Z_1) \cup \bigcup_{i \neq X \setminus \bar{D}} \{t'_i \times \sigma_1(t'_i)\}, \quad \ldots \quad (2.5)$$

$V(D, \bar{\beta}) = D$ and from the formal equality (2.1) and equalities (2.2), (2.5) follows that

$$Z_i \bar{\beta} = P_i \bar{\beta} = Z_i,$$

$$D \bar{\beta} = P_i \bar{\beta} \cup P_i \bar{\beta} = Z_i \cup \bar{\bar{D}} = \bar{D},$$

$$\alpha = \delta \circ \bar{\beta} = (Y_i^0 \times Z_i \bar{\beta}) \cup (Y_i^0 \times \bar{\bar{D}} \bar{\beta}) =$$

$$= (Y_i^0 \times Z_i) \cup (Y_i^0 \times \bar{D}) = X \times Z \notin B$$

since $V(X^*, \alpha) = \{Z_i\} \neq D$.

If $\sigma_2 = \begin{pmatrix} P_0 & P_1 \\ Z_0 & \bar{D} \end{pmatrix}$ and $\sigma_3$ is mapping of the set $X \setminus \bar{D}$ in the semilattice $D$. So, if

$$\bar{\beta} = (P_i \times Z_i) \cup (\bar{P}_i \times \bar{D}) \cup \bigcup_{i \neq X \setminus \bar{D}} \{t'_i \times \sigma_3(t'_i)\}, \quad \ldots \quad (2.6)$$

then $V(D, \bar{\beta}) = D$. From the formal equality (2.1) and equalities (2.2), (2.6) we have:

$$Z_i \bar{\beta} = P_i \bar{\beta} = Z_i,$$

$$D \bar{\beta} = P_i \bar{\beta} \cup \bar{P}_i \bar{\beta} = Z_i \cup \bar{D} = \bar{D},$$

$$\alpha = \delta \circ \bar{\beta} = (Y_i^0 \times Z_i \bar{\beta}) \cup (Y_i^0 \times \bar{D} \bar{\beta}) =$$

$$= (Y_i^0 \times Z_i) \cup (Y_i^0 \times \bar{D}) = X \times \bar{D} \notin B$$

But, the equality $\alpha = \delta$ contradict the condition, that $\delta \in B \setminus \{D\} \setminus \{\alpha\}$. That is in this case $\alpha \notin B$.

If $\sigma_3 = \begin{pmatrix} P_0 & P_1 \\ \bar{D} & Z_1 \end{pmatrix}$ and $\sigma_4$ is a mapping of the set $X \setminus \bar{D}$ in the semilattice $D$. So, if

$$\bar{\beta} = (P_i \times \bar{D}) \cup (P_i \times Z_i) \cup \bigcup_{i \neq X \setminus \bar{D}} \{t'_i \times \sigma_4(t'_i)\}, \quad \ldots \quad (2.7)$$

then $V(D, \bar{\beta}) = D$. From the formal equality (2.1) and equalities (2.2), (2.7) we have:

$$Z_i \bar{\beta} = P_i \bar{\beta} = D,$$

$$D \bar{\beta} = P_i \bar{\beta} \cup \bar{P}_i \bar{\beta} = \bar{D} \cup \bar{Z}_i = D,$$

$$\alpha = \delta \circ \bar{\beta} = (Y_i^0 \times Z_i \bar{\beta}) \cup (Y_i^0 \times D \bar{\beta}) =$$

$$= (Y_i^0 \times \bar{D}) \cup (Y_i^0 \times D) = X \times \bar{D} \notin B$$

since $V(X^*, \alpha) = \{\bar{D}\} \neq D$.

Let $\sigma_4 = \begin{pmatrix} P_0 & P_1 \\ \bar{D} & \bar{D} \end{pmatrix}$. If $\sigma_4$ be a mapping of the set $X \setminus \bar{D}$ on the set $D \setminus \{\bar{D}\} = \{Z_i\}$, then

$$\bar{\beta} = ((P_i \cup R) \times \bar{D}) \cup \bigcup_{i \neq X \setminus \bar{D}} \{t'_i \times \sigma_4(t'_i)\}, \quad \ldots \quad (2.8)$$

$V(D, \bar{\beta}) = D$ and from the formal equality (2.1) and equalities (2.2), (2.8) follows that

$$Z_i \bar{\beta} = P_i \bar{\beta} = \bar{D},$$

$$D \bar{\beta} = P_i \bar{\beta} \cup \bar{P}_i \bar{\beta} = \bar{D} \cup \bar{D} = \bar{D},$$

$$\alpha = \delta \circ \bar{\beta} = (Y_i^0 \times Z_i \bar{\beta}) \cup (Y_i^0 \times \bar{D} \bar{\beta}) =$$

$$= (Y_i^0 \times \bar{D}) \cup (Y_i^0 \times \bar{D}) = X \times \bar{D} \notin B$$

since $V(X^*, \alpha) = \{\bar{D}\} \neq D$. 

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So, from the cases b) follows that \( B \) is a set external elements of semigroup \( B_{\alpha}(D) \) since the mappings 
\( \sigma_{i} - \sigma_{i} \) are all mappings of the set \( \{P_{i}, P_{i} \} \) in the semilattice \( D \).

Lemma 2.1 is proved.

**Corollary 2.1.** Let \( D = \{\emptyset, D\} \in \Sigma_{i}(X, 2) \) and \( B = \{ \alpha \in B_{\alpha}(D) | V(X, \alpha) = D \} \). Then the following statements are true:

1. If \( |X \setminus D| \geq 1 \) and \( Z_i = \emptyset \), then \( \alpha = X \times D \) do not generating by elements of the set \( B \);
2. If \( X = \tilde{D} \) and \( Z_i \neq \emptyset \), then \( \alpha = X \times Z_i \) do not generating by elements of the set \( B \);
3. If \( X = \tilde{D} \) and \( Z_i = \emptyset \), then \( \alpha = \emptyset \) and \( \alpha = X \times \tilde{D} \) do not generating by elements of the set \( B \).

**Proof.** Let \( |X \setminus D| \geq 1 \), \( Z_i = \emptyset \) and \( \delta, \beta \in B \). Then quasinormal representation of a binary relation \( \delta \) has a form 
\[
\delta = (Y^{\delta}_{1} \times \emptyset) \cup (Y^{\delta}_{0} \times D),
\]
where \( Y^{\delta}_{1}, Y^{\delta}_{0} \notin \{ \emptyset \} \) and
\[
\delta \circ \beta = (Y^{\delta}_{1} \times \emptyset \beta) \cup (Y^{\delta}_{0} \times \tilde{D} \beta) = (Y^{\delta}_{1} \times \emptyset) \cup (Y^{\delta}_{0} \times \tilde{D} \beta) \neq X \times \tilde{D},
\]
since \( Y^{\delta}_{1} \neq \emptyset \).

Therefore, if \( |X \setminus D| \geq 1 \) and \( Z_i = \emptyset \), then \( \alpha = X \times D \) do not generating by elements of the set \( B \).

The statement 1) of the Corollary 2.1 is proved.

Let \( X = D \) and \( Z_i \neq \emptyset \). If \( \delta \) and \( \beta \) are such elements of the set \( B \), that \( \delta \circ \beta = X \times Z_i \), then quasinormal representation of a binary relation \( \delta \) has a form 
\[
\delta = (Y^{\delta}_{1} \times Z_i) \cup (Y^{\delta}_{0} \times \tilde{D}),
\]
where \( Y^{\delta}_{1}, Y^{\delta}_{0} \notin \{ \emptyset \} \) and
\[
\delta \circ \beta = (Y^{\delta}_{1} \times Z_i \beta) \cup (Y^{\delta}_{0} \times \tilde{D} \beta) = (Y^{\delta}_{1} \times Z_i) \cup (Y^{\delta}_{0} \times X \beta) = X \times Z_i,
\]
i.e. \( Z_i \beta = X \beta = Z_i \). Of the equality \( X \beta = Z_i \) follows that \( t \beta = Z_i \) for all \( t \in X \) since \( Z_i \) be smallest element of the semilattice \( D \). So, the equality \( \beta = X \times Z_i \) is true. Last equality contradict the condition \( \beta \in B \) since \( V(X, \beta) = \{Z_i\} \neq D \). Therefore, binary relation \( \alpha = X \times Z_i \) do not generating by elements of the set \( B \).

The statement 2) of the Corollary 2.1 is proved.

Let \( X = D \), \( Z_i = \emptyset \). If \( \alpha = \delta \circ \beta \) for some \( \delta, \beta \in B \), then quasinormal representations of binary relations \( \delta \) and \( \beta \) has a form
\[
\delta = (Y^{\delta}_{1} \times \emptyset) \cup (Y^{\delta}_{0} \times D) = (Y^{\delta}_{1} \times \emptyset) \cup (Y^{\delta}_{0} \times X),
\]
\[
\beta = (Y^{\beta}_{1} \times \emptyset) \cup (Y^{\beta}_{0} \times D) = (Y^{\beta}_{1} \times \emptyset) \cup (Y^{\beta}_{0} \times X),
\]
where \( Y^{\delta}_{1}, Y^{\delta}_{0}, Y^{\beta}_{1}, Y^{\beta}_{0} \notin \{ \emptyset \} \) since \( V(X, \delta) = V(X, \beta) = D \) (\( \delta, \beta \in B \)). So, we have:
\[
\alpha = \delta \circ \beta = \left( (Y^{\delta}_{1} \times \emptyset) \cup (Y^{\delta}_{0} \times X) \right) \circ \left( (Y^{\beta}_{1} \times \emptyset) \cup (Y^{\beta}_{0} \times X) \right) = \left( (Y^{\delta}_{1} \times \emptyset) \circ (Y^{\beta}_{1} \times \emptyset) \cup (Y^{\delta}_{0} \times \emptyset) \circ (Y^{\beta}_{0} \times \emptyset) \right) \cup \left( (Y^{\delta}_{1} \times \emptyset) \circ (Y^{\beta}_{1} \times X) \cup (Y^{\delta}_{0} \times \emptyset) \circ (Y^{\beta}_{0} \times X) \right) = \emptyset \cup \emptyset \cup \emptyset \cup \emptyset \times \emptyset = \emptyset
\]
since \( X \times Y^{\beta}_{1} = Y^{\beta}_{1} \neq \emptyset \) and \( X \times Y^{\beta}_{0} = Y^{\beta}_{0} \neq \emptyset \). Bat from the equalities \( \alpha = \emptyset \), \( \alpha = \delta \) or \( \alpha = X \times \tilde{D} \), \( \alpha = \delta \) respectively follows that \( \alpha \notin B \) and \( \alpha \notin B \) since \( \delta \in B \) by preposition, i.e. \( \alpha = \emptyset \) and \( \alpha = X \times \tilde{D} \) do not generating by elements of the set \( B \).

The statement 3) of the Corollary 2.1 is proved.

The Corollary 2.1 is proved.
Theorem 2.1. Let $D = \{Z_i, \bar{D}\} \in \Sigma(X, 2)$. If $B = \{\alpha \in B_x(D) \mid V \{X^*, \alpha\} = D\}$, then the following statements are true:

a) if $|X \setminus \bar{D}| \geq 1$ and $Z_i \neq \emptyset$, then $B$ is irreducible generating set for the semigroup $B_x(D)$;

b) if $|X \setminus \bar{D}| \geq 1$ and $Z_i = \emptyset$, then $B \cup \{X \times \bar{D}\}$ is irreducible generating set for the semigroup $B_x(D)$.

c) if $X = \bar{D}$ and $Z_i \neq \emptyset$, then $B \cup \{X \times Z_i\}$ is irreducible generating set for the semigroup $B_x(D)$;

d) if $X = \bar{D}$ and $Z_i = \emptyset$, then $B \cup \{\emptyset, X \times \bar{D}\}$ is irreducible generating set for the semigroup $B_x(D)$.

Proof. Let $|X \setminus \bar{D}| \geq 1$. Then binary relation $\alpha$ has a quasinormal representation of the form $\alpha = (Y^*_i \times Z_i) \cup (\bar{Y}_i^* \times \bar{D})$, where $Y^*_i = \emptyset$ or $\bar{Y}_i^* = \emptyset$ ($V \{X^*, \alpha\} \neq D$ since $\alpha \in B$).

Let $Y_i^* = \emptyset$. Then $\alpha = X \times \bar{D}$ and for any $\delta = (Y^*_i \times Z_i) \cup (\bar{Y}_i^* \times \bar{D}) \in B$ and for $\beta = (Z_i \times \bar{D}) \cup (X \times Z_i \times Z_i)$ we have $\beta \in B$ since $Z_i \neq \emptyset$ by preposition and $X \setminus \bar{D} \supset X \times Z_i \neq \emptyset$. So, the following equalities hold: $\delta \circ \beta = (Y^*_i \times Z_i) \cup (\bar{Y}_i^* \times \bar{D}) = (Y^*_i \times Z_i) \cup (X \times Z_i \times Z_i) = (Y^*_i \times Z_i) \cup (X \times Z_i \times Z_i)$

since $Z_i \subset D$ by preposition.

Let $Y_i^* = \emptyset$. Then $\alpha = X \times \bar{D}$ and for any $\delta = (Y^*_i \times Z_i) \cup (\bar{Y}_i^* \times \bar{D}) \in B$ and for $\beta = (\bar{D} \times Z_i) \cup (X \setminus \bar{D} \times \bar{D})$ we have $\beta \in B$ since $X \setminus D \neq \emptyset$ ($|X \setminus \bar{D}| \geq 1$). So, the following equalities are true: $\delta \circ \beta = (Y^*_i \times Z_i) \cup (\bar{Y}_i^* \times \bar{D}) = (Y^*_i \times Z_i) \cup (X \times \bar{D} \times \bar{D})$.

$\cup (Y^*_i \times \bar{D}) \circ (X \setminus \bar{D} \times \bar{D}) = (Y^*_i \times Z_i) \cup (X \times \bar{D} \times \bar{D}) = (Y^*_i \times Z_i) \cup (X \times \bar{D} \times \bar{D}) = X \times Z_i = \alpha$

since $\bar{D} \supset Z_i \neq \emptyset$ by preposition.

The statement a) of the Theorem 2.1 is proved.

Let $|X \setminus \bar{D}| \geq 1$ and $Z_i = \emptyset$. Then for any $\delta = (Y^*_i \times \bar{D}) \cup (\bar{Y}_i^* \times \bar{D}) \in B$ and for $\beta = (\bar{D} \times \bar{D}) \cup (X \setminus \bar{D} \times \bar{D})$ we have $\beta \in B$ since $X \setminus D \neq \emptyset$ ($|X \setminus \bar{D}| \geq 1$). So, the following equalities are true: $\delta \circ \beta = (Y^*_i \times \bar{D}) \cup (\bar{Y}_i^* \times \bar{D}) = (Y^*_i \times \bar{D}) \cup (X \setminus \bar{D} \times \bar{D})$

$\cup (Y^*_i \times \bar{D}) \circ (X \setminus \bar{D} \times \bar{D}) = (Y^*_i \times \bar{D}) \cup (X \setminus \bar{D} \times \bar{D}) = (Y^*_i \times \bar{D}) \cup (X \setminus \bar{D} \times \bar{D}) = \emptyset \cup (Y^*_i \times \bar{D}) \cup (X \setminus \bar{D} \times \bar{D}) = \emptyset = \alpha$.

Now, the statement b) of the Theorem 2.1 immediately follows from the statement 1) of the Corollary 2.1.

Let $X = \bar{D}$ and $Z_i \neq \emptyset$. If $\delta = (Y^*_i \times Z_i) \cup (\bar{Y}_i^* \times \bar{D})$ be any element of the set $B$ and $\beta = (Z_i \times \bar{D}) \cup (X \times Z_i \times Z_i)$. It is easy to see, that $\beta \in B$ and
\[ \delta \circ \beta = ((Z_i \times Z_i) \cup ((X \setminus Z_i) \times D)) \cup ((Z_i \times D) \cup ((X \setminus Z_i) \times Z_i)) = \]
\[ = (Z_i \times Z_i) \cup ((X \setminus Z_i) \times D) \cup ((X \setminus Z_i) \times Z_i) \cup ((X \setminus Z_i) \times Z_i) \]
\[ = (Z_i \times Z_i) \cup ((X \setminus Z_i) \times D) \cup ((X \setminus Z_i) \times Z_i) \cup ((X \setminus Z_i) \times Z_i) \]
\[ = (Z_i \times D) \cup ((X \setminus Z_i) \times Z_i) \cup ((X \setminus Z_i) \times Z_i) = \]
\[ = Z_i \times Z_i \cup (X \setminus Z_i) \times D) = X \times D \]

since \( Z_i \subset \tilde{D} \) by preposition.

Now, the statement \( c \) of the Theorem 2.1 immediately follows from the statement \( 2 \) of the Corollary 2.1.

The statement \( d \) of Theorem 2.1 immediately follows from the statement \( 3 \) of the Corollary 2.1.

The Theorem 2.1 is proved.

**Theorem 2.2.** Let \( X \) be finite a set, \( D = \{Z_i, \tilde{D}\} \in \Sigma_i (X, 2). \) If \( |X| = n, \) then for the number of the irreducible generated set \( B' \) of the semigroup \( B_x (D) \) following statements are true:

a) if \( |X \setminus \tilde{D}| \geq 1 \) and \( Z_i \neq \emptyset, \) then \( |B'| = 2^n - 2; \)

b) if \( |X \setminus \tilde{D}| \geq 1 \) and \( Z_i = \emptyset \) or \( X = \tilde{D} \) and \( Z_i \neq \emptyset, \) then \( |B'| = 2^n - 1; \)

d) if \( X = \tilde{D} \) and \( Z_i = \emptyset, \) then \( |B'| = 2^n. \)

**Proof.** It is well known, that if \( B \) is all external elements of the semigroup \( B_x (D) \) and \( B' \) be any generated set for the \( B_x (D), \) then \( B \subseteq B'. \) By Lemma 2.1 The set \( B = \{\alpha \in B_x (D) \mid V (X', \alpha) = D\} \) is a set external elements of the semigroup \( B_x (D). \) It is easy to see, that \( B = B_x(D) \setminus \{X \times Z_i, X 	imes \tilde{D}\}. \) Of this follows that \( |B| = 2^n - 2 \) (see 1.1).

By statement \( a \) of the Theorem 2.1 follows that \( B = B', \) i.e. \( |B'| = 2^n - 2. \)

By statement \( b \) and \( c \) of the Theorem 2.1 follows that \( B' = B \cup \{X \times \tilde{D}\} \) or \( B' = B \cup \{X \times Z_i\}, \) i.e. \( |B'| = 2^n - 2 + 1 = 2^n - 1. \)

By statement \( d \) of the Theorem 2.1 follows that \( B' = B \cup \{X \times Z_i, X \times \tilde{D}\}, \) i.e. \( |B'| = (2^n - 2) + 2 = 2^n. \)

Theorem 2.2 is proved.

**Example 2.1.** Let \( X = \{1,2,3\} \) and \( D = \{\{1\}, \{1,2\}\}, \) i.e. \( |X \setminus \tilde{D}| = 1 \) and \( Z_i = \{1\} \neq \emptyset \) (see statement \( a \)).

Then \( B_x(D) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \), where

\[ \alpha_1 = \{(1,1),(2,1),(3,1)\}, \quad \alpha_2 = \{(1,1),(2,1),(3,1),(3,2)\}, \]
\[ \alpha_3 = \{(1,1),(2,1),(2,2),(3,1)\}, \quad \alpha_4 = \{(1,1),(2,1),(2,2),(3,1),(3,2)\}, \]
\[ \alpha_5 = \{(1,1),(1,2),(2,1),(3,1)\}, \quad \alpha_6 = \{(1,1),(1,2),(2,1),(3,1),(3,2)\}, \]
\[ \alpha_7 = \{(1,1),(1,2),(2,1),(2,2),(3,1)\}, \quad \alpha_8 = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}. \]

In this case we have \( B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\} \) and

\[
\begin{array}{cccccccc}
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\alpha_3 & \alpha_4 & \alpha_3 & \alpha_1 & \alpha_2 & \alpha_5 & \alpha_6 & \alpha_7 \\
\alpha_4 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_1 & \alpha_2 & \alpha_6 & \alpha_7 \\
\alpha_5 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_7 & \alpha_6 \\
\alpha_6 & \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_7 & \alpha_6 \\
\alpha_7 & \alpha_8 & \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_8 & \alpha_6 \\
\end{array}
\]
So, we have that $B = \{ \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \}$ is irreducible generated set for the semigroup $B_x(D)$.

**Example 2.2.** Let $X = \{1,2,3\}$ and $D = \{\emptyset, \{1,2\}\}$, i.e. $|X \setminus \bar{D}| = 1$ and $Z_1 = \emptyset$ (see statement $b)$). Then $B_x(D) = \{\alpha_1, \alpha_2, \ldots, \alpha_8\}$, where

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In this case we have $B = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\} \cup \{\alpha_8\}$ and

So, we have that $B = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\} \cup \{\alpha_8\}$ is irreducible generated set for the semigroup $B_x(D)$.

**Example 2.3.** Let $X = \{1,2\}$ and $D = \{\{1\}, \{1,2\}\}$, i.e. $X = \bar{D} = \{1\}$ and $Z_1 = \emptyset$ (see statement $b)$). Then $B_x(D) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$\alpha_1 = \{(1,1),(2,1)\}$,
$\alpha_2 = \{(1,1),(2,1),(2,2)\}$,
$\alpha_3 = \{(1,1),(1,2),(2,1)\}$,
$\alpha_4 = \{(1,1),(1,2),(2,1),(2,2)\}$.

In this case we have $B = \{\alpha_1, \alpha_2\} \cup \{\alpha_3\}$ and

So, we have that $B = \{\alpha_2, \alpha_3\} \cup \{\alpha_1\}$ is irreducible generated set for the semigroup $B_x(D)$.

**Example 2.4.** Let $X = \{1,2\}$ and $D = \{\emptyset, \{1,2\}\}$, i.e. $X = \bar{D} = \{1,2\}$ and $Z_1 = \emptyset$ (see statement $d)$). Then $B_x(D) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$\alpha_1 = \emptyset$, $\alpha_2 = \{(2,1),(2,2)\}$,
$\alpha_3 = \{(1,1),(1,2)\}$,
$\alpha_4 = \{(1,1),(1,2),(2,1),(2,2)\}$.

In this case we have $B = \{\alpha_2, \alpha_3\} \cup \{\alpha_1, \alpha_4\}$ and

So, we have that $B = \{\alpha_2, \alpha_3\} \cup \{\alpha_1, \alpha_4\}$ is irreducible generated set for the semigroup $B_x(D)$.
So, we have that $B = \{ \alpha_2, \alpha_3 \} \cup \{ \alpha_4, \alpha_5 \}$ is irreducible generated set for the semi group $B_X (D)$.

**REFERENCES**


