On Some Properties of Regular Elements of Complete Semi groups Defined by Semi lattices of the Class \( \Sigma 4(X, 8) \)

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**ABSTRACT:** In this article, we prove that all regular elements \( R_n \) of the complete semi group \( B_x(D) \) defined by an \( X \) – semi lattice \( D \) of the class \( \Sigma 4(X, 8) \) are subsemigroups of this semigroup.

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**I. INTRODUCTION**

Let \( X \) be an arbitrary nonempty set, \( D \) be an \( X \) – semi lattice of unions, i.e. such a nonempty set of subsets of the set \( X \) that is closed with respect to the set-theoretic operations of unification of elements from \( D \) . \( f \) be an arbitrary mapping of the set \( X \) in the set \( D \) . To each mapping \( f \) we put into correspondence a binary relation \( \alpha_f \) on the set \( X \) that satisfies the condition

\[
\alpha_f = \bigcup_{x \in X} \{(x) \times f(x)\}.
\]

The set of all such \( \alpha_f \) ( \( f : X \to D \) ) is denoted by \( B_x(D) \) . It is easy to prove that \( B_x(D) \) is a semi group with respect to the operation of multiplication of binary relations, which is called a complete semi group of binary relations defined by an \( X \) – semi lattice of unions \( D \) .

We denote by \( \emptyset \) an empty binary relation or an empty subset of the set \( X \) . The condition \( (x, y) \in \alpha \) will be written in the form \( x \alpha y \) . Further, let \( x, y \in X \) , \( Y \subseteq X \) , \( \alpha \in B_x(D) \) , \( \emptyset \neq D' \subseteq D \) , \( D = \bigcup Y \) and \( t \in D' \) . We denote by the symbols \( y\alpha \) , \( Y\alpha \) , \( V(D, \alpha) \) , \( X' \) , \( V(X', \alpha) \) and \( D'_t \) the following sets:

\[
\begin{align*}
y\alpha &= \{x \in X \mid y\alpha x\}, \\
Y\alpha &= \bigcup_{y \in Y} y\alpha, \\
V(D, \alpha) &= \{Y\alpha \mid Y \in D\}, \\
X' &= \{Y \mid \emptyset \neq Y \subseteq X\}, \\
V(X', \alpha) &= \{Y\alpha \mid \emptyset \neq Y \subseteq X\}, \\
D'_t &= \{Z' \mid t \in Z'\}.
\end{align*}
\]

We use the symbol \( \Lambda(D, D') \) to denote the exact lower bound of the set \( D' \) in the semilattice \( D \) .

**Definition 1.** We say that the complete \( X \) – semilattice of unions \( D \) is an \( XI \) – semilattice of unions if it satisfies the following two conditions:

a) \( \Lambda(D, D) \in D \) for any \( t \in D ; \)

b) \( Z = \bigcup_{t \in Z} \Lambda(D, D) \) for any nonempty element \( Z \) of the semilattice \( D \) . (see [1] and [2]).

Let \( X \) and \( \Sigma_4(X, 8) \) be respectively an arbitrary nonempty set and the class of \( X \) – semilattices of unions, where each element is isomorphic to some \( X \) – semilattice of unions \( D = \{Z_1, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \tilde{D}\} \) that satisfies the conditions

\[
\begin{align*}
\Lambda(D, D) &\in D \\
Z &= \bigcup_{t \in Z} \Lambda(D, D)
\end{align*}
\]
An $X$ – semilattice that satisfies conditions (1) is shown in Fig. 1.

![Fig.1](image)

For the complete semigroups of binary relations defined by semilattices of the class $\Sigma_4(X,8)$ the following statements (see [7, 8]) are well known.

**Lemma 1.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \in \Sigma_4(X,8)$. Then the following sets exhaust all $XI$ – subsemilattices of the semilattice $D$:

1) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, \{Z_1\}, \{D\}\}$ (see diagram 1 in Fig. 2);

2) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, \{Z_1, Z_7\}, \{Z_1, Z_6\}, \{Z_1, Z_5\}, \{Z_1, D\}, \{Z_1, Z_4\}, \{Z_1, Z_3\}\}$;

3) $\{Z_7, Z_6, \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, \{Z_1, D\}, \{Z_1, Z_7\}, \{Z_1, Z_6\}, \{Z_1, Z_5\}, \{Z_1, Z_4\}, \{Z_1, Z_3\}\}$ (see diagram 2 in Fig. 2);

4) $\{Z_7, Z_6, Z_5, Z_4, Z_3, D\}, \{Z_7, Z_6, Z_5, Z_4, Z_3, D\}$ (see diagram 3 in Fig. 2);

5) $\{Z_7, Z_6, Z_5, Z_4, Z_3, D\}, \{Z_7, Z_6, Z_5, Z_4, Z_3, D\}$ (see diagram 4 in Fig. 2);

6) $\{Z_7, Z_6, Z_5, Z_4, Z_3, D\}$ (see diagram 5 in Fig. 2);

7) $\{Z_7, Z_6, Z_5, Z_4, Z_3, D\}$ (see diagram 6 in Fig. 2);

8) $\{Z_7, Z_6, Z_5, Z_4, Z_3, D\}$ (see diagram 7 in Fig. 2);

9) $\{Z_7, Z_6, Z_5, Z_4, Z_3, D\}$ (see diagram 8 in Fig. 2).

Fig. 2 gives the diagrams of all $XI$ – subsemilattices of $D$.

![Fig.2](image)

**Definition 2.** Let $\beta \in B_x(D)$. If $\beta \circ \delta \circ \beta = \beta$ for some $\delta \in B_x(D)$, then a binary relation $\beta$ is called a regular element of the semi group $B_x(D)$. 

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Definition 3. The one-to-one mapping \( \varphi \) between the complete \( \mathcal{X} \)-semilattices of unions \( D' \) and \( D^* \) is called a complete isomorphism if the condition

\[
\varphi(\cup_{D_i}) = \bigcup_{T \in D_i} \varphi(T')
\]

is fulfilled for each nonempty subset \( D_i \) of the semilattice \( D' \) (see [1], Definition 6.3.2).

Definition 4. Let \( \alpha \) be some binary relation of the semi group \( B_{\mathcal{X}}(D) \). We say that a complete isomorphism \( \varphi \) between the complete semilattices of unions \( Q \) and \( D' \) is a complete \( \alpha \)-isomorphism if

a) \( Q = V(D, \alpha) \);

b) \( \varphi(\emptyset) = \emptyset \) for \( \emptyset \in V(D, \alpha) \) and \( \varphi(T) \alpha = T \) for any \( T \in V(D, \alpha) \) (see [1], Definition 6.3.3)).

Definition. Let \( D \) be an arbitrary complete \( \mathcal{X} \)-semilattice of unions, \( \alpha \in B_{\mathcal{X}}(D) \) and \( Y^a_T = \{ x \in X \mid x \alpha = T \} \). If

\[
V[\alpha] = \begin{cases} 
V\left(X^*, \alpha\right), & \text{if } \emptyset \not\subseteq D, \\
V\left(X^*, \alpha\right), & \text{if } \emptyset \not\subseteq V\left(X^*, \alpha\right), \\
V\left(X^*, \alpha\right) \cup \{\emptyset\}, & \text{if } \emptyset \not\subseteq V\left(X^*, \alpha\right) \text{ and } \emptyset \in D,
\end{cases}
\]

then it obviously follows that any binary relation \( \alpha \) of the semi group \( B_{\mathcal{X}}(D) \) can always be represented in the form \( \alpha = \bigcup_{\alpha \in \Omega} (Y^a_T \times T) \).

In the sequel we will call such a representation of a binary relation \( \alpha \) quasinormal.

Note that for a quasinormal representation of a binary relation \( \alpha \), it is not all sets \( Y^a_T \) that may differ from an empty set. But for such a representation the following conditions are always fulfilled:

a) \( Y^a_T \cap Y^a_{T'} = \emptyset \) for any \( T, T' \in D \) and \( T \neq T' \);

b) \( X = \bigcup_{\alpha \in \Omega} Y^a_T \) (see [1], 1.11).

Theorem 1. Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}\} \in \Sigma(X, 8) \). Then a binary relation \( \alpha \) of the semigroup \( B_{\mathcal{X}}(D) \) with a quasinormal representation in one of the forms given below is a regular element of this semigroup if there exists a complete \( \alpha \)-isomorphism \( \varphi \) of the semilattice \( V(D, \alpha) \) on some subsemilattice \( D' \) of the semilattice \( D \) that satisfies at least one of the following conditions:

a) \( \alpha = X \times T \) for some \( T \in D \);

b) \( \alpha = (Y_T \times T) \cup (Y^a_T \times T) \) for some \( T, T' \in D \) and \( T \subseteq T' \) and \( Y^a_T, Y^a_T \not\subseteq \emptyset \), which satisfies the conditions:

\( Y^a_T \not\subseteq \varphi(T) \), \( Y^a_T \cap \varphi(T') \neq \emptyset \);

c) \( \alpha = (Y_T \times T) \cup (Y^a_T \times T) \cup (Y^a_{T'} \times T^*) \) for some \( T, T', T^* \in D \) and \( T \subseteq T' \subseteq T^* \), and \( Y^a_T, Y^a_T, Y^a_T \not\subseteq \emptyset \), which satisfies the conditions:

\( Y^a_T \not\subseteq \varphi(T) \), \( Y^a_T \cup Y^a_T \not\subseteq \varphi(T') \), \( Y^a_T \cap \varphi(T') \neq \emptyset \), \( Y^a_T \cap \varphi(T^*) \neq \emptyset \);

d) \( \alpha = (Y_T \times Z) \cup (Y^a_T \times T) \cup (Y^a_{T'} \times T^*) \cup (Y^a_{T''} \times \bar{D}) \) for some \( T, T', T'' \in D \), \( Z \in T \subseteq T' \subseteq T'' \), \( Y^a_T, Y^a_T, Y^a_T \not\subseteq \emptyset \), and \( Y^a_T \not\subseteq \varphi(T) \), \( Y^a_T \cup Y^a_T \not\subseteq \varphi(T') \), \( Y^a_T \cup Y^a_T \not\subseteq \varphi(T^*) \), \( Y^a_T \cap \varphi(T'') \neq \emptyset \), \( Y^a_T \cap \varphi(T') \neq \emptyset \);

e) \( \alpha = (Y_T \times T) \cup (Y^a_T \times T) \cup (Y^a_T \times Z) \cup (Y^a_{T''} \times (T' \times Z)) \) for some \( T, T', T'' \in D \), \( T \subseteq T' \subseteq T'' \), \( T' \not\subseteq \emptyset \), \( Z \not\subseteq \emptyset \), \( Y^a_T, Y^a_T \not\subseteq \emptyset \), and \( Y^a_T \not\subseteq \varphi(T) \), \( Y^a_T \cup Y^a_T \not\subseteq \varphi(Z) \), \( Y^a_T \cap \varphi(T') \neq \emptyset \), \( Y^a_T \cap \varphi(T'') \neq \emptyset \), \( Y^a_T \cap \varphi(Z) \neq \emptyset \).
\[ f) \quad \alpha = (Y_4^w \times Z_7 \cup (Y_7^w \times Z_6) \cup (Y_6^w \times Z_4) \cup (Y_4^w \times Z_1) \cup (Y_1^w \times Z_6) \cup (Y_6^w \times \bar{D}) \] for some \( Y^w_7, Y^w_6, Y^w_4, Y^w_1, Y^w_6, Y^w_6 \in \emptyset \) which satisfies the conditions: \( Y^w_1 \cup Y^w_6 \equiv \varphi(Z_7) \), \( Y^w_6 \cup Y^w_6 \equiv \varphi(Z_4) \), \( Y^w_6 \cap \varphi(Z_6) \neq \emptyset \), \( Y^w_6 \cap \varphi(Z_6) = \emptyset \), \( Y^w_6 \cap \varphi(\bar{D}) \neq \emptyset \).

\[ g) \quad \alpha = (Y_7^w \times Z_7) \cup (Y_6^w \times Z_6) \cup (Y_1^w \times Z_1) \cup (Y_6^w \times Z_6) \cup (Y_6^w \times \bar{D}) \] for some \( Y^w_7, Y^w_6, Y^w_1, Y^w_6 \in \emptyset \) which satisfies the conditions: \( Y^w_1 \cup Y^w_6 \equiv \varphi(Z_7) \), \( Y^w_6 \cup Y^w_6 \equiv \varphi(Z_4) \), \( Y^w_6 \cup Y^w_1 \cup Y^w_6 \equiv \varphi(Z_6) \), \( Y^w_6 \cap \varphi(Z_6) \neq \emptyset \), \( Y^w_6 \cap \varphi(Z_6) = \emptyset \), \( Y^w_6 \cup \varphi(\bar{D}) \neq \emptyset \).

\[ h) \quad \alpha = (Y_7^w \times Z_7) \cup (Y_6^w \times Z_6) \cup (Y_1^w \times Z_1) \cup (Y_6^w \times Z_6) \cup (Y_6^w \times \bar{D}) \] for some \( Y^w_7, Y^w_6, Y^w_1, Y^w_6 \in \emptyset \) which satisfies the conditions: \( Y^w_7 \equiv \varphi(Z_7) \), \( Y^w_7 \equiv \varphi(Z_4) \), \( Y^w_7 \cup Y^w_1 \equiv \varphi(Z_6) \), \( Y^w_7 \cap \varphi(Z_6) \neq \emptyset \), \( Y^w_7 \cap \varphi(Z_6) = \emptyset \), \( Y^w_7 \cap \varphi(\bar{D}) \neq \emptyset \).

**Theorem 2.** Let \( D = [Z_7, Z_6, Z_1, Z_6, Z_1, Z_1, \bar{D}] \in \Sigma_4 (X, 8) \). Then the set \( R_0 \) of all regular elements of the semigroup \( B_4 (D) \) is a subsemigroup of this semigroup.

Proof. Let \( \alpha, \beta \in B_4 (D) \). Then for a binary relation \( \alpha \) of the semigroup \( B_4 (D) \) that has a quasinormal representation of the form

\[ \alpha = (Y_7^w \times Z_7) \cup (Y_6^w \times Z_6) \cup (Y_1^w \times Z_1) \cup (Y_6^w \times Z_6) \cup (Y_6^w \times \bar{D}) \] we have

\[ V(D, \alpha \circ \beta) \subseteq V(D, \beta) \] \[ \ldots (3) \]

and

\[ \alpha \circ \beta = (Y_7^w \times Z_7) \cup (Y_6^w \times Z_6) \cup (Y_1^w \times Z_1) \cup (Y_6^w \times Z_6) \cup (Y_6^w \times \bar{D}) \] \[ \ldots (4) \]

Therefore

\[ V(D, \alpha \circ \beta) = \{ Z, Z_7, Z_6, Z_6, Z_1, Z_6, Z_1, Z_1, \bar{D}, \bar{D}, \} \ldots (5) \]

Note that the mapping \( \varphi_{\alpha \beta} : V(D, \alpha) \rightarrow V(D, \alpha \circ \beta) \), which satisfies the condition \( \varphi_{\alpha \beta} (Z) \) is \( Z \) for any element \( Z \) of the set \( V(D, \alpha) \), is the monotone mapping of the semilattice \( V(D, \alpha) \) on the semilattice \( V(D, \alpha \circ \beta) \), i.e., \( \varphi_{\alpha \beta} \) is the mapping for which the condition \( Z \subseteq Z' \) always implies the inclusion \( Z \beta \subseteq Z' \beta \) for each \( Z, Z' \in V(D, \alpha) \). From inclusion (3) we have that the mapping \( \varphi_{\alpha \beta} \) is the monotone mapping of the semilattice \( V(D, \alpha) \) in the semilattice \( V(D, \beta) \), i.e., if the semilattice \( V(D, \alpha) \) is finite and its diagram is well known, then the diagram of the semilattice \( V(D, \alpha \circ \beta) \) is a subdiagram of the diagram of the semilattice \( V(D, \beta) \).

Now let the binary relations \( \alpha \) and \( \beta \) be any regular elements of the semigroup \( B_4 (D) \). Then \( V(D, \alpha) \) and \( V(D, \beta) \) are \( XI \) semilattices and by Theorem 1 the diagrams of all subsemilattices of \( D \) are given in Fig. 2. From these diagrams it follows that the semilattice \( V(D, \alpha) \) has the smallest element. From equalities (3) and (5) it follows that the semilattice \( V(D, \alpha \circ \beta) \) also has the smallest element. This fact immediately implies that the mapping \( \varphi_{\alpha \beta} \) is the monotone mapping of the semilattice \( V(D, \alpha) \) in the semilattice \( V(D, \beta) \).

We consider the following case.

1) The diagram of the semilattice \( V(D, \beta) \) has form 8 in Fig. 2. From the definition of this semilattice \( D \) it follows that we have a unique subsemilattice \( D' = [Z_7, Z_6, Z_6, Z_1, \bar{D}] \) of the semilattice \( D \), the diagram of which has form 8 in Fig. 2 (see Fig. 3). Any subsemilattice of this semilattice has a diagram of the form shown in Fig. 4.
By assumption, we have \( V(D, \alpha \circ \beta) \subseteq V(D, \beta) \). So, the diagram of the semilattice \( V(D, \alpha \circ \beta) \) can be one of the diagrams shown in Fig. 4. But by Theorem 1 the semilattice \( V(D, \alpha \circ \beta) \) has the smallest element. From this it follows that the semilattice \( V(D, \alpha \circ \beta) \) can have one of diagrams 1 – 9 given in Fig. 4.

Let us prove that the diagram of the semilattice \( V(D, \alpha \circ \beta) \) is always different from diagram 9 in Fig. 4. Indeed, diagram 9 is a subdiagram of diagram 3. Therefore, by the definition of the semilattice \( D \), the following equalities are fulfilled:

\[
V(D, \alpha) = \{Z_7, Z_6, Z_5, Z_4, Z_3, D\} \quad \text{and} \quad V(D, \alpha \circ \beta) = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \subseteq V(D, \beta).
\]

Suppose that there exists some regular element \( \beta \) for which \( Z \beta = Z' \beta \) for some \( Z \beta, Z' \beta \in V(D, \alpha \circ \beta) \) and the other five elements are pairwise disjoint subsets of the set \( X \).

We consider the following cases:

a) \( Z, Z', Z'' \in D \), \( Z \subset Z' \subset Z'' \), \( Z \beta, Z' \beta, Z'' \beta \in V(D, \alpha \circ \beta) \) and \( Z \beta = Z' \beta = Z'' \beta \). Then \( Z \beta = Z' \beta = Z'' \beta \), i.e., in this case we have \( V(D, \alpha \circ \beta) \leq 4 \). The inequality \( V(D, \alpha \circ \beta) \leq 4 \) contradicts the equality \( V(D, \alpha \circ \beta) = 5 \).

So, we have \( Z \beta \neq Z' \beta, Z \beta \neq Z'' \beta, Z \beta \neq \bar{D} \beta, Z \beta \neq \bar{D} \beta \).

b) \( Z \beta = Z' \beta \). By the definition of the semilattice \( D \) we have \( Z \beta \cup Z' \beta = Z_1 \) and \( Z \beta \supseteq Z_2 \). From this it follows that \( Z \beta \cup Z_2 \beta = Z_3 \beta \), \( Z_2 \beta \supseteq Z_3 \beta = Z_4 \beta \), and \( Z_2 \beta \cup Z_3 \beta = Z_4 \beta \). So, \( V(D, \alpha \circ \beta) \leq 4 \). The inequality \( V(D, \alpha \circ \beta) \leq 4 \) contradicts the equality \( V(D, \alpha \circ \beta) = 5 \). Therefore \( Z \beta \neq Z_2 \beta \).

c) \( Z_3 \beta = Z_4 \beta \). By the definition of the semilattice \( D \) we have \( Z_3 \beta \cup Z_4 \beta = Z_5 \), \( Z_4 \beta \supseteq Z_5 \beta = Z_6 \beta \), and \( Z_4 \beta \cup Z_5 \beta = Z_6 \beta \). So, \( V(D, \alpha \circ \beta) \leq 4 \). The inequality \( V(D, \alpha \circ \beta) \leq 4 \) contradicts the equality \( V(D, \alpha \circ \beta) = 5 \). Therefore \( Z_4 \beta \neq Z_5 \beta \).

d) \( Z_5 \beta = Z_6 \beta \). By the definition of the semilattice \( D \) we have \( Z_5 \beta \cup Z_6 \beta = \bar{D} \beta \), \( Z_5 \beta \supseteq Z_7 \beta = \bar{D} \beta \), and \( Z_5 \beta \cup Z_7 \beta = \bar{D} \beta \). So, \( V(D, \alpha \circ \beta) \leq 4 \). The inequality \( V(D, \alpha \circ \beta) \leq 4 \) contradicts the equality \( V(D, \alpha \circ \beta) = 5 \). Therefore \( Z_5 \beta \neq Z_7 \beta \).

e) \( Z_6 \beta = Z_7 \beta \). By the definition of the semilattice \( D \) we have \( Z_6 \beta \cup Z_7 \beta = \bar{D} \beta \). Thus we have \( Z_6 \beta \cup Z_7 \beta = \bar{D} \beta \), \( Z_6 \beta \supseteq Z_8 \beta \), and \( Z_6 \beta \cup Z_8 \beta = \bar{D} \beta \). So, \( V(D, \alpha \circ \beta) \leq 4 \). The inequality \( V(D, \alpha \circ \beta) \leq 4 \) contradicts the equality \( V(D, \alpha \circ \beta) = 5 \). Therefore \( Z_6 \beta \neq Z_7 \beta \).
f) \( Z_\alpha \beta = Z_\beta \). Since five elements of the semilattice \( V(D, \alpha \circ \beta) \) are pairwise disjoint subsets of the set \( X \), the following inclusions are true:

\[
Z_\beta \subseteq Z_n \beta \subseteq Z_\alpha \beta \subseteq \bar{D} \beta, \ Z_\beta \subseteq Z_n \beta \subseteq Z_\alpha \beta \subseteq \bar{D} \beta,
\]

i.e., the diagram of the semilattice \( V(D, \alpha \circ \beta) \) is a chain if the elements \( Z_\beta \) and \( Z_n \beta \) are correlated with set-theoretical inclusions or the diagram of the semilattice \( V(D, \alpha \circ \beta) \) has form 7 shown in Fig. 4 if \( Z_\beta \setminus Z_\alpha \beta \neq \emptyset \) and \( Z_\beta \setminus Z_\alpha \beta \neq \emptyset \). This contradicts our proposition that the diagram of the semilattice \( V(D, \alpha \circ \beta) \) has form 9 shown in Fig. 4. So, we have \( Z_\beta \neq Z_\alpha \beta \).

g) \( Z_\beta = \bar{D} \beta \) or \( Z_\beta = \bar{D} \beta \). Since five elements of the semilattice \( V(D, \alpha \circ \beta) \) are pairwise different subsets of the set \( X \), we have

\[
Z_\beta \subseteq Z_n \beta \subseteq Z_\alpha \beta \subseteq \bar{D} \beta, \ Z_\beta \subseteq Z_n \beta \subseteq Z_\alpha \beta \subseteq \bar{D} \beta
\]

or

\[
Z_\beta \subseteq Z_n \beta \subseteq Z_\alpha \beta \subseteq \bar{D} \beta, \ Z_\beta \subseteq Z_n \beta \subseteq Z_\alpha \beta \subseteq \bar{D} \beta.
\]

i.e., the diagram of the semilattice \( V(D, \alpha \circ \beta) \) is a chain if the elements \( Z_\beta \) and \( Z_n \beta \) are correlated with set-theoretical inclusions or the diagram of the semilattice \( V(D, \alpha \circ \beta) \) has form 6 in Fig. 4 if \( Z_\beta \setminus Z_\alpha \beta \neq \emptyset \) and \( Z_\beta \setminus Z_\alpha \beta \neq \emptyset \). This contradicts our proposition that the diagram of the semilattice \( V(D, \alpha \circ \beta) \) has form 9 in Fig. 4. So, we have \( Z_\beta \neq \bar{D} \beta \) and \( Z_n \beta \neq \bar{D} \beta \).

h) \( Z_\alpha \beta = Z_\beta \) or \( Z_\alpha \beta = Z_\beta \) or \( Z_\alpha \beta = Z_\beta \). By the definition of the semilattice \( D \) we have \( Z_\alpha \cup Z_\beta = Z_\alpha \cup Z_\beta = Z_\beta \). From these conditions it follows that \( Z_\beta \setminus Z_\alpha \beta = Z_\beta \) and \( Z_\alpha \beta \setminus Z_\beta = Z_\beta \setminus Z_\beta = \bar{D} \beta \). From this and by the proposition we have \( Z_\beta = Z_\beta = Z_\beta \), \( Z_\beta = Z_\beta = \bar{D} \beta \), \( Z_\alpha \beta = Z_\beta = \bar{D} \beta \). So, \( \|V(D, \alpha \circ \beta)\| \leq 4 \). This contradicts our proposition since \( \|V(D, \alpha \circ \beta)\| = 5 \). Therefore \( Z_\alpha \beta \neq Z_\alpha \beta \) and \( Z_\alpha \beta \neq Z_\beta \).

From the cases a)–h) it follows that the diagram of the semilattice \( V(D, \alpha \circ \beta) \) may never have form 9 in Fig. 4. Thus the diagram of the semilattice \( V(D, \alpha \circ \beta) \) may be one of diagrams 1–8 given in Fig. 4. By virtue of Theorems 11.6.1, 11.6.3 and 11.7.2 (See [1]), they are \( XI \)–semilattices of unions. Therefore, from Theorem 14.20.1 (see [1]) it follows that the binary relation \( \alpha \circ \beta \) is a regular element of the semigroup \( B_X(D) \).

2) If the diagram of the semilattice \( V(D, \beta) \) has form 7 in Fig. 2, then the diagram of any subsemilattice of the semilattice \( V(D, \beta) \) may have one of forms 1–7 of shown in Fig. 5.

![Fig. 5](image)

Since the semilattice \( V(D, \alpha \circ \beta) \) has the smallest element, the diagram of the semilattice \( V(D, \alpha \circ \beta) \) may have one of forms 1–6 in Fig. 5. By Theorems 11.6.1 and 11.6.3 (see [1]), they are \( XI \)–semilattices of unions. Therefore, from Theorem 14.20.1 (see [1]) it follows that the binary relation \( \alpha \circ \beta \) is a regular element of the semigroup \( B_X(D) \).

3) The diagram of the semilattice \( V(D, \beta) \) has form 6 shown in Fig. 2. Then the diagram of any subsemilattice of the semilattice \( V(D, \beta) \) may have one of forms 1–8 shown in Fig. 2.
Since the semilattice $V(D, \alpha \circ \beta)$ has the smallest element, the diagram of the semilattice $V(D, \alpha \circ \beta)$ may have one of forms 1–6 given in Fig. 6. By Theorems 11.6.1 and 11.6.3 (see [1]) they are $XI$–semilattices of unions. Therefore, from the Theorem 14.20.1 (see [1]) it follows that the binary relation $\alpha \circ \beta$ is a regular element of the semigroup $B_\xi(D)$.

4) The diagram of the semilattice $V(D, \beta)$ has form 5 given in Fig. 2. Then the diagram of any subsemilattice of the semilattice $V(D, \beta)$ may have one of forms 1–5 shown in Fig. 2.

Since the semilattice $V(D, \alpha \circ \beta)$ has the smallest element, the diagram of the semilattice $V(D, \alpha \circ \beta)$ may have one of forms 1–4 shown in Fig. 7. By Theorems 11.6.1 and 11.6.3 (see [1]) it follows that they are $XI$–semilattices of unions. Therefore, from the Theorem 14.20.1 (see [1]) it follows that the binary relation $\alpha \circ \beta$ is a regular element of the semigroup $B_\xi(D)$.

5) The diagram of the semilattice $V(D, \beta)$ has forms 1–4 given in Fig. 2. Then the diagram of any subsemilattice of the semilattice $V(D, \beta)$ is a finite chain. By Theorem 11.6.1 (see [1]) it follows that they are $XI$–semilattices of unions. Therefore, from Theorem 14.20.1 (see [1]) it follows that the binary relation $\alpha \circ \beta$ is a regular element of the semigroup $B_\xi(D)$.

The theorem is proved.

REFERENCES


[8] Diasamidze Ya., Bakuridze A. Idempotent elements of complete semi groups of binary relations defined by semilattices of the class $\Sigma_4(X,8)$ (to appear).
[9] Diasamidze Ya., Bakuridze A. Regular elements of complete semi groups of binary relations defined by semilattices of the class $\Sigma_i (X, 8)$ (to appear).