On Rg-Compact Spaces

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Abstract: The purpose of this paper is to offer some more properties of rg-compact spaces such as the finite union of rg-compact sets, and then we studied its connection with rg-closed sets and rg-Hausdorff spaces.

Key Words: rg-compact spaces, rg-Hausdorff spaces, rg-closed sets.

I. INTRODUCTION

Regular generalized closed sets have been introduced and studied by Palaniappar and Chandra [3], Chawlit [2] introduced the concept of rg-Hausdorff spaces, Al-Shibani [1] introduced and investigated rg-compact spaces. Throughout this paper $(X, \tau)$ (or simply $X$) represents a topological space. For any subset $A$ of $X$, cl $A$ and int $A$ denote the closure of $A$ and the interior of $A$, respectively.

Definition 1 A subset $A$ of $X$ is said to be:
1. Regular open (briefly, r-open) (resp. regular closed (briefly, r-closed)) if $A = \text{int(cl} A \text{)}$ (resp. $A = \text{cl(int} A \text{)}$) [4].
2. rg-closed if cl $A \subseteq U$ whenever $A \subseteq U$, where $U$ is r-open.
It is said to be rg-open if $X \setminus A$ is rg-closed (equivalently $F \subseteq \text{int} A$ whenever $F \subseteq A$ and $F$ is r-closed) [3].

Remark 1 It is easy to see that r-open $\implies$ open $\implies$ rg-open

Definition 2 A topological space $X$ is said to be rg-Hausdorff if whenever $x$ and $y$ are distinct points of $X$ there are disjoint rg-open sets $U$ and $V$ with $x \in U$ and $y \in V$ [2].

Remark 2 It is easy to see that Hausdorff $\implies$ rg-Hausdorff.

Definition 3 A topological space $X$ is called regular generalized compact (briefly, rg-compact) if every rg-open cover of $X$ has a finite subcover [1].

Remark 3 It is easy to see that rg-compact $\implies$ compact.

Example 1 Every finite subset of a topological space is rg-compact.

Proof. Let $A = \{a_1, \ldots, a_n\}$ be finite subset of a topological space $X$, let $\{U_{a_i}\}$ be rg-open cover of $A$ which means $A \subseteq \bigcup_{a \in A} U_a$.

Then each element of $A$ belongs to at least one of the sets $\{U_{a_i}\}$, hence $a_1 \in U_{a_1}, \ldots, a_n \in U_{a_n}$

Thus $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$

Which is a finite subcover for $A$. So $A$ is rg-compact.

Theorem 1 Let $A, B$ be rg-compact subsets of a topological space $X$. Then $A \cup B$ is rg-compact.
Proof. Let \( \{U_\alpha\}_{\alpha \in I} \) be rg-open cover of \( A \cup B \), i.e., \( A \cup B \subseteq \bigcup_{\alpha \in I} U_\alpha \). Since \( A \) and \( B \) are rg-compact, so we can choose finite subcover, say \( A \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_m} \), \( B \subseteq U_{\alpha_{m+1}} \cup \ldots \cup U_{\alpha_{m+k}} \). Hence \( A \cup B \subseteq \bigcup_{i=1}^{m+k} U_{\alpha_i} \).

Therefor \( A \cup B \) is rg-compact. \( \square \)

Corollary 1 Any finite union of rg-compact sets are rg-compact.

Theorem 2 Each rg-compact subset of a rg-Hausdorff space is rg-closed.

Proof. Let \( A \) be rg-compact subset of a Hausdorff space \( X \). We will prove that \( X \setminus A \) is rg-open and hence \( A \) is rg-closed.

Let \( x \in X \setminus A \), for each \( y \in A \) there exists disjoint rg-open sets \( U_y \) and \( V_x \) with \( y \in U_y \) and \( x \in V_x \).

The collection \( \{U_y: y \in A\} \) is rg-open cover of \( A \).

Theorem 3 Let \( X \) be rg-compact, rg-Hausdorff space, \( A \) subset of \( X \) is rg-compact if and only if it is rg-closed.

Proof. Obvious from Theorem (2) and Theorem 4.2 in [3]. \( \square \)

Theorem 4 If \( A \) is rg-compact, \( F \) is rg-closed subset of a topology space \( X \), then \( A \cap F \) is rg-compact.

Proof. Let \( \{U_\alpha\} \) be rg-open cover of \( A \cap F \), i.e., \( A \cap F \subseteq \bigcup_{\alpha \in I} U_\alpha \), then \( A \subseteq \bigcup_{\alpha \in I} U_\alpha \cup (X \setminus F) \).

Since \( F \) is rg-closed subset, so \( X \setminus F \) is rg-open subset, therefore \( \{U_\alpha\}, X \setminus F \) is rg-open cover of \( A \).

Theorem 5 The following conditions are equivalent:
1. \( X \) is rg-compact
2. Every family \( \{F_\alpha\} \) where \( I \) is index set of rg-closed subsets of \( X \) with \( \bigcap_{\alpha \in I} F_\alpha = \emptyset \), contains a finite subsets \( \{F_{\alpha_1}, \ldots, F_{\alpha_k}\} \) such that \( \bigcap_{i=1}^k F_{\alpha_i} = \emptyset \).

Proof. 1. \( \Rightarrow \) 2. Let \( \bigcap_{\alpha \in I} F_\alpha = \emptyset \), then by taking complement

\[
X \setminus \bigcap_{\alpha \in I} F_\alpha = X
\]

\[
\Rightarrow \bigcup_{\alpha \in I} (X \setminus F_\alpha) = X
\]
So $\{X \setminus F_{\alpha}\}_{\alpha \in I}$ is rg-open cover of $X$, since $X$ is rg-compact, it has a finite subcover, thus

$$X = \bigcup_{i=1}^{k} (X \setminus F_{\alpha_i})$$

Now taking complement

$$\phi = X \setminus \bigcup_{i=1}^{k} (X \setminus F_{\alpha_i})$$

$$= \bigcap_{i=1}^{k} F_{\alpha_i}$$

2. $\Rightarrow 1.$ Let $\{U_\alpha\}_{\alpha \in I}$ be rg-open cover of $X$.

i.e. $X = \bigcup_{\alpha \in I} U_\alpha$, so $X \setminus U_\alpha, \alpha \in I$ are rg-closed subsets which have an empty intersection.

By hypothes it contains a finite subset $\{X \setminus U_{\alpha_1}, ..., X \setminus U_{\alpha_k}\}$ such that $\bigcap_{i=1}^{k} (X \setminus U_{\alpha_i}) = \phi$.

By taking complement

$$X \setminus \bigcap_{i=1}^{k} X \setminus U_{\alpha_i} \supseteq X \setminus \phi$$

$$\Rightarrow \bigcup_{i=1}^{k} U_{\alpha_i} = X$$

Thus $X$ is rg-compact.

**Theorem 6** A topological space $X$ is rg-compact if and only if every family of rg-closed subset of $X$ which has the finite intersection property, has a nonempty intersection.

**Proof.** From the previous theorem we can take the negation of 2 i.e. for every rg-closed sets $F_{\alpha}, \alpha \in I$ such that

$$\bigcap_{i=1}^{k} F_{\alpha_i} \neq \phi \implies \bigcap_{\alpha \in I} F_{\alpha} \neq \phi.$$

So the proof is completed.

**REFERENCES**


