Integral Type of Contractions under f-weak Reciprocally Continuous

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Abstract— The purpose of this paper is to establish some fixed point theorems under the integral type of contractive conditions by using the new notion of f-weakly reciprocally continuous mappings for a pair of self mappings and cyclic mappings of a G-metric space. These results improve and extend some earlier results in the literature of fixed point theory, in particular, the recent results of Shatanawi et al. and Bilgili et al.

Index Terms—Fixed point, G-Metric Space, f-weakly reciprocally continuous, compatible, g-compatible, f-compatible.

I. INTRODUCTION

The concept of a G-metric space was introduced by Mustafa and Sims in [1], wherein, the authors discussed the topological properties of this space and proved the analogue of the Banach contraction principle in the context of G-metric space. Since then, many interesting results have been obtained in the framework of G-metric space by various authors (see [2]-[5]). Integral type of contractive condition in fixed point theorems was introduced by Branciari[6]. Later, Aydi[3] extended these results to the class of G-metric spaces, while Shatanawi[4] initiated the study of -maps in G-metric spaces. In [2], Shatanawi et al. have combined the concepts of -maps and integral type of contractions and proved some fixed point results in G-metric space. The concept of cyclic mappings was introduced by Kirk et al. and later improved by Karapinar et al.(refer[13]-[16])

In this paper, we define f-weakly reciprocally continuous mappings which is a generalization of f-reciprocally continuous mappings and we employ this notion to obtain a fixed point theorem for a pair of self mappings of a G-metric space, which generalizes the results of Shatanawi et al[2]. Also we prove another fixed point theorem for a cyclic mapping of a G-metric space by using the integral type of contractions, which generalizes the results of Bilgili et al.[12].

The following are the basic definitions needed in the main results.

Definition 1.1:[1] Let X be a nonempty set and be a function satisfying the following properties

(G1) \( G(x, y, z) = 0 \) if \( x = y = z \)

(G2) \( 0 < G(x, x, y) \) for all \( x, y \in X \) with \( xy \).

(G3) \( G(x, x, y) \leq G(x, y, z) \) for all \( x, y, z \in X \) with \( xz \).

(G4) \( G(x, y, z) = G(x, z, y) = G(y, z, x) \) (symmetry in all three variables).

(G5) \( G(x, x, z) \leq G(x, x, a) + G(a, y) \) for all \( x, y, z, a \in X \) (rectangle inequality).

Then the function G is called a generalized metric or, more specifically, a G-metric on X, and the pair \((X, G)\) is called a G-metric space.

Definition 1.2:[1] Let \((X, G)\) be a G-Metric space. A sequence \( \{x_n\} \) is called a G-Cauchy sequence if for any \( \varepsilon > 0 \), there is a natural number \( N \) such that \( G(x_n, x_m, x_l) < \varepsilon \) for all \( n, m, l \geq N \).

A G-Metric space \((X, G)\) is said to be G-complete if every G-Cauchy sequence is G-convergent in X.

In their initial paper Mustafa and Sims [1] also proved the following proposition.

Proposition: Let \((X, G)\) be a G-metric space. Then, for any \( x, y, z, a \in X \), it follows that

1. If \( G(x, y, z) = 0 \), then \( x = y = z \).
2. \( G(x, y, z) \leq G(x, y, y) + G(x, x) \).
3. \( G(x, y, y') \leq 2G(y, x) \).
4. \( G(x, y, z) \leq G(x, a, z) + G(a, y) \)

5. \( G(x, y, z) \leq \frac{2}{3} G(x, y, a) + G(x, a, z) + G(a, y) \)

6. \( G(x, y, z) \leq \frac{1}{2} G(x, a, a) + G(y, a, z) + G(z, a) \)

**Definition 1.3:** Let \( f \) and \( g \) be self mappings on a \( G \)-Metric space \((X, G)\). The mappings \( f \) and \( g \) are called
1. Compatible [8] if \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) : \text{for some } t \in X \).
2. \( g \)-compatible [9] if \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) : \text{for some } t \in X \).
3. \( f \)-compatible [9] if \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) : \text{for some } t \in X \).
4. Compatible of type (P) [10] if \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) : \text{for some } t \in X \).
5. \( f \)-reciprocally continuous [11] if \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) : \text{for some } t \in X \).

**Definition 1.4:** [13] Let \( A_1, A_2, ..., A_m \) be nonempty subsets of a \( G \)-Metric space \((X, G)\). Then the mapping \( T : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{n} A_i \) is called a cyclic mapping if \( T(A_i) \subseteq A_i \), where \( A_{m+1} = A_1 \).

In 1977, Mathkowski [7] introduced the \( \Phi \) map as the following:

Let be the set of auxiliary functions such that \( \Phi : [0, \infty) \to [0, 1] \) is a nondecreasing function satisfying \( \lim_{n \to \infty} \Phi^n(t) = \text{for all } t \in (0, 1) \). If \( \Phi \in \Phi \), then \( \Phi \) is called a \( \phi \)-map. Further \( \Phi(t) < \text{for all } t \in (0, 1) \) and \( \Phi(0) = 0 \).

Let be the set of functions \( \Phi : [0, \infty) \to [0, 1] \) where \( \Phi \) is a lebesgue integrable mapping, summable, non-negative and for each \( \epsilon, a, b > 0 \),

\[
\int_{0}^{\epsilon} \Phi(t) dt \quad \text{and} \quad \int_{a}^{b} \Phi(t) dt \leq \int_{[a, b]} \Phi(t) dt + \int_{\epsilon}^{b} \Phi(t) dt.
\]

**II. MAIN RESULT**

We now define \( f \)-weakly reciprocally continuous mappings which is a generalized notion of \( f \)-reciprocal continuity.

**Definition 2.1:** Let \( f \) and \( g \) be self mappings on a \( G \)-Metric space \((X, G)\). Then the mappings \( f \) and \( g \) are called \( f \)-weakly reciprocally continuous if \( \lim_{n \to \infty} f(x_n) = ft \) or \( \lim_{n \to \infty} g(x_n) = gt \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) : \text{for some } t \in X \).

**Example:** Let \( X = [1, 10] \) and \( G : X \times X \times X \to [0, \infty) \) defined by \( G(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\} \)

where \( d \) is the usual metric on \( X \). Then, \( (X, G) \) is a \( G \)-Metric space. Define \( f, g : X \to X \) by

\[
f(x) = \begin{cases} \frac{x+1}{2} & \text{if } x < 6 \\ 3 & \text{if } x = 3 \\ \frac{10}{3} & \text{if } x > 6 \\ \frac{2}{3} & \text{if } x \geq 6 
\end{cases}
\]

and \( g(x) = \begin{cases} 3 & \text{if } x = 3 \\ 10 & \text{if } x < 3 \text{ and } 3 \leq x < 6 \\ \frac{2}{3} & \text{if } x \geq 6 
\end{cases} \)

let \( \{x_n\} = \{6 + \frac{1}{n}\} \) be a sequence in \( X \). Then \( f(x_n) \to 3 \) and \( g(x_n) = \frac{10}{3} \to 3. \)

\( fgx_n = f(3 + \frac{1}{n}) \to 3 \) and \( ggx_n = g(3 + \frac{1}{n}) \to 10. \) Thus \( \lim_{n \to \infty} fgx_n = f3 \) but \( \lim_{n \to \infty} ggx_n = g3 \).

Hence \( f \) and \( g \) are \( f \)-weakly reciprocally continuous mappings but not \( f \)-reciprocally continuous.

**Remark:** If \( f \) and \( g \) are \( f \)-reciprocally continuous, then they are obviously \( f \)-weakly reciprocally continuous but, the converse is not true as shown in the above example. Thus, \( f \)-weak reciprocal continuity is a proper generalization of \( f \)-reciprocal continuity.

**Theorem 2.2:** Let \( f \) and \( g \) be \( f \)-weakly reciprocally continuous self mappings on a \( G \)-complete \( G \)-Metric space \((X, G)\) with \( fX \subseteq gX \). Let \( \Phi \in \Phi \) satisfying
For all \( x, y, z \in X \), where \( \phi \in \Psi \) and 
\[ L(x, y, z) = \max \{ G(gx, gy, fz), G(gx, gz, fy), G(gy, fx, fz), G(gy, fz, gx), G(gz, fz, fx), G(gz, fx, gy) \} \]
If \( f \) and \( g \) are either compatible or \( g \)-compatible or \( f \)-compatible or compatible of type (P), then \( f \) and \( g \) have a unique common fixed point.

**Proof:** Let \( x_0 \) be any point in \( X \). Since \( fX \subseteq gX \), there exists a sequence of points \( x_0, x_1, x_2, \ldots, x_n, \ldots \) such that \( x_{n+1} = f x_n \) in the pre-image under \( g \) of \( fX \), i.e., \( f x_0 = g x_1, f x_1 = g x_2, \ldots, f x_n = g x_{n+1}, \ldots \)

Let \( \{ y_n \} \) be a sequence in \( X \) such that \( y_n = f x_n = g x_{n-1} \) for \( n = 0, 1, 2, \ldots \)

Assume that \( G(y_{n+1}, y_{n+1}, y_n) > 0 \) for all \( n = 0, 1, 2, \ldots \) otherwise we obtain \( y_{n+1} = y_n \). Therefore \( y_{n+1} = y_n \) for \( n = 0, 1, 2, \ldots \). Now, we claim that \( \{ y_n \} \) is a \( G \)-cauchy sequence in \( X \).

Consider 
\[ \int_0^L \phi(t) dt = \int_0^{L(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt \]
(2)

where 
\[ L(x_{n+1}, x_{n+1}, x_n) = \max \{ G(y_{n+1}, y_{n+1}, y_n), G(y_n, y_{n+1}, y_{n+1}), G(y_{n+1}, y_{n+1}, y_n) \} \]
If \( L(x_{n+1}, x_{n+1}, x_n) = G(y_{n+1}, y_{n+1}, y_n) \) then from (2) we have 
\[ \int_0^L \phi(t) dt \leq \phi \left( \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt \right) < \phi \left( \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt \right) \]

a contradiction. If \( L(x_{n+1}, x_{n+1}, x_n) = G(y_n, y_{n+1}, y_{n+1}) \) then from (2) we have 
\[ \int_0^L \phi(t) dt \leq \phi \left( \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt \right) \leq \cdots \leq \phi \left( \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt \right) \]

Let \( \varepsilon > 0 \) be given. Since \( \phi \left( \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt \right) \to 0 \) as \( n \to \infty \), there exists an integer \( l_0 \) such that 

\[ \phi \left( \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt \right) < \frac{1}{3} (\varepsilon - \phi (\varepsilon)) \quad \forall n \geq l_0 \]

Therefore 
\[ \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt < \frac{1}{3} (\varepsilon - \phi (\varepsilon)) \quad \forall n \geq l_0 \]

Similarly, if \( L(x_{n+1}, x_{n+1}, x_n) = G(y_n, y_n, y_{n+1}) \) then we obtain 
\[ \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt < \frac{1}{3} (\varepsilon - \phi (\varepsilon)) \quad \forall n \geq l_1 \]

Let \( l = \max \{ l_0, l_1 \} \). Then, 
\[ \int_0^{G(y_{n+1}, y_{n+1}, y_n)} \phi(t) dt < \frac{1}{3} (\varepsilon - \phi (\varepsilon)) \quad \forall n \geq l \]
(3)

Let \( k, n \in \mathbb{N} \cup \{0\} \). Then we claim that 
\[ \int_0^{G(k^n, y_{n+1}, y_n)} \phi(t) dt < \varepsilon \quad \forall k \geq n \geq l \]
(4)

We prove (4) by induction on \( k \). Note that (4) holds for \( k = n + 1 \) by using (3). Assume that (4) holds for \( k = m \), i.e., 
\[ \int_0^{G(k^m, y_{n+1}, y_n)} \phi(t) dt < \varepsilon \quad \forall m \geq n \geq l \]
(5)

For \( k = m + 1 \), we have
where, \( L(x_{m+1},x_{m+1},x_{n+1}) = \max \{G(y_n,y_{n+1},y_{n+1}) + G(y_{m+1},y_{m+1},y_{n+1})\}, \)

If \( L(x_{m+1},x_{m+1},x_{n+1}) = G(y_n,y_{n+1},y_{n+1}) + G(y_{m+1},y_{m+1},y_{n+1}) \), then from (6) we have

\[
\int_0^\infty \varphi(t) \, dt \leq \int_0^\infty \varphi(t) \, dt + \int_0^\infty \frac{\varphi(t) \, dt}{3}.
\]

Therefore by induction on \( k \), we conclude that (4) holds good for all \( k \geq n \geq 1 \). Since \( \varepsilon \) is arbitrary, we have

\[
G(y_n,y_{n+1}) \to 0 \quad \text{as} \quad n,m \to \infty.
\]

Therefore \( G(y_n,y_{n+1}) \to 0 \) as \( m \to \infty \). That is \( \{y_n\} \) is a \( G \)-cauchy sequence in \( X \). Since \( X \) is \( G \)-complete there exists a point \( t \) in \( X \) such that

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_{n+1} = t.
\]

**Case I:** Let \( f \) and \( g \) be compatible. Then \( \lim_{n \to \infty} G(f x_n,g x_n,f x_{n+1}) = 0 \).

(f,g) are \( f \)-weakly reciprocally continuous mappings implies \( f g x_n \to ft \) or \( g g x_n \to gt \).

First, let \( f g x_n \to ft \). Since \( f X \subseteq g X \), there exists a point \( z \in X \) such that \( ft = gz \). Therefore \( f g x_n \to gz \) which implies \( f x_n \to gz \) by the compatibility of \( f \) and \( g \). Also \( f g x_{n+1} = f f x_n \to gz \).

Consider

\[
\int_0^\infty \varphi(t) \, dt \leq \varphi \left( \int_0^\infty \varphi(t) \, dt \right)
\]

where

\[
L(z,z,f x_n) = \max \{G(gz,fz,f x_n),G(gz,gz,ff x_n),G(gz,gz,gf x_n),G(gz,fz,ff x_n)\}.
\]

Letting \( n \to \infty \) in the above equation we obtain,

\[
\int_0^\infty \varphi(t) \, dt \leq \varphi \left( \int_0^\infty \varphi(t) \, dt \right)
\]

where

\[
L(z,z,t) = \max \{G(gz,fz,gz),G(gz,fz,fz)\}.
\]

If \( L(z,z,t) = G(gz,fz,gz) \), then from (7) we obtain
which implies \( gz = fz \).

If \( g(z, z, t) = G(gz, fz, fz) \), then from (7) we obtain

\[
\int_0^t \varphi(t) dt \leq \varphi \left( \int_0^t \varphi(t) dt \right) \leq 2 \int_0^t \varphi(t) dt
\]

which gives \( gz = fz \).

By Compatibility of \( f \) and \( g \) we have \( ffz = fgz = gfz = ggz \). Consider

\[
\int_0^t \varphi(t) dt \leq \varphi \left( \int_0^t \varphi(t) dt \right)
\]

where \( L(z, z, fz) = G(fz, fz, fz) \).

Therefore

\[
\int_0^t \varphi(t) dt \leq \varphi \left( \int_0^t \varphi(t) dt \right) \leq 0.
\]

Hence \( fz = ffz = gfz \). Thus \( fz \) is a common fixed point of \( f \) and \( g \).

Next, suppose that \( gg x_n \rightarrow gt \), then by compatibility \( fg x_n \rightarrow gt \).

This implies \( gg x_{n+1} = gf x_n \rightarrow gt \) and \( fg x_{n+1} = ff x_n \rightarrow gt \).

Using the similar arguments as above, we can show that \( gt = ft = fft = gft \). Thus \( ft \) is a common fixed point of \( f \) and \( g \).

Case II: Let \( f \) and \( g \) be \( g \)-compatible. Then \( \lim_{n \rightarrow \infty} G(ff x_n, gf x_n, gf x_n) = 0 \).

\( (f, g) \) are \( f \)-weakly reciprocally continuous mappings implies \( fg x_n \rightarrow ft \) or \( gg x_n \rightarrow gt \).

First, let \( fg x_n \rightarrow ft \). Since \( fx \subseteq gx \), there exists a point \( z \in X \) such that \( ft = gz \). Therefore \( fg x_n \rightarrow gz \) and \( fg x_{n+1} = ff x_n \rightarrow gz \) which implies \( gf x_n \rightarrow gz \), by \( g \)-compatibility. Using the similar arguments as in case I, we can show that \( gz = fz = ffz = gfz \). Thus \( fz \) is a common fixed point of \( f \) and \( g \).

Next, suppose that \( gg x_n \rightarrow gt \), then \( gg x_{n+1} = gf x_n \rightarrow gt \). This implies \( ff x_n \rightarrow gt \) and \( fg x_{n+1} = ff x_n \rightarrow gt \). Using the similar arguments as in case I, we can show that \( gt = ft = fft = gft \). Thus \( ft \) is a common fixed point of \( f \) and \( g \).

Case III: Let \( f \) and \( g \) be \( f \)-compatible. Then \( \lim_{n \rightarrow \infty} G(gg x_n, gg x_n, gg x_n) = 0 \).

\( (f, g) \) are \( f \)-weakly reciprocally continuous mappings implies \( fg x_n \rightarrow ft \) or \( gg x_n \rightarrow gt \).

First, let \( fg x_n \rightarrow ft \). Since \( fx \subseteq gx \), there exists a point \( z \in X \) such that \( ft = gz \). Therefore \( fg x_n \rightarrow gz \) which implies \( gg x_n \rightarrow gz \), by \( f \)-compatibility. Therefore \( gg x_{n+1} = gf x_n \rightarrow gz \) and \( fg x_{n+1} = ff x_n \rightarrow gz \). Using the similar arguments as in case I, we can show that \( gz = fz = ffz = gfz \). Thus \( fz \) is a common fixed point of \( f \) and \( g \).

Next, suppose that \( gg x_n \rightarrow gt \), then by \( f \)-compatibility \( fg x_n \rightarrow gt \). This implies \( gg x_{n+1} = gf x_n \rightarrow gt \) and \( fg x_{n+1} = ff x_n \rightarrow gt \). Using the similar arguments as in case I, we can show that \( gt = ft = fft = gft \). Thus \( ft \) is a common fixed point of \( f \) and \( g \).

Case IV: Let \( f \) and \( g \) be compatible of type (\( P \)). Then \( \lim_{n \rightarrow \infty} G(ff x_n, gg x_n, gg x_n) = 0 \).

\( (f, g) \) are \( f \)-weakly reciprocally continuous mappings implies \( fg x_n \rightarrow ft \) or \( gg x_n \rightarrow gt \).

First, let \( fg x_n \rightarrow ft \). Since \( fx \subseteq gx \), there exists a point \( z \in X \) such that \( ft = gz \). Therefore \( fg x_n \rightarrow gz \) and \( x_{n+1} = ff x_n \rightarrow gz \). Therefore \( gg x_{n+1} = gf x_n \rightarrow gz \). Using the similar arguments as in case I, we can show that \( gz = fz = ffz = gfz \). Thus \( fz \) is a common fixed point of \( f \) and \( g \).

Next, suppose that \( gg x_n \rightarrow gt \), then by compatibility of type (\( P \)) \( ff x_n \rightarrow gt \). This implies \( gg x_{n+1} = gf x_n \rightarrow gt \) and \( ff x_n = fg x_{n+1} \rightarrow gt \). Using the similar arguments as in case I, we can show that
Thus \( ft = g ft = g ft \). Thus \( ft \) is a common fixed point of \( f \) and \( g \). In all the cases, the uniqueness of the fixed point can be proved easily.

We now illustrate this theorem by giving two examples.

**Example 1:** Let \( X = [0,10] \) and \( G : X \times X \times X \to [0,\infty) \) defined by
\[
G(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \text{ and } \max \{ x,y,z \} \text{ in all other cases. Then } (X,G) \text{ is a } G\text{-Metric space.} \\
\end{cases}
\]
Let \( f \) and \( g \) be two self maps on \( X \) defined by
\[
f(x) = \begin{cases} \frac{x}{2} & \text{if } x \leq 5 \\
2 - \frac{x}{5} & \text{if } x > 5 \\
\end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{x}{5} & \text{if } x \leq 5 \\
\frac{x}{2} & \text{if } x > 5 \\
\end{cases}
\]
Define \( \phi : [0,\infty) \to [0,\infty) \) by \( \phi(t) = \frac{2t}{5} \). Here \( f \) and \( g \) are \( G \)-compatible and \( f \)-weakly reciprocally continuous. To see this, consider a sequence \( \{ x_n \} = \left\{ \frac{1}{n} \right\} \) for all \( n \). Then \( f x_n \to 0, g x_n \to 0 \) and \( G(f x_n, g x_n, f x_n) \to 0 \).

Example 2:** Let \( X = [1,10] \) and \( G : X \times X \times X \to [0,\infty) \) defined by
\[
G(x,y,z) = \max \{ d(x,y), d(y,z), d(z,x) \} \text{ where } d \text{ is the usual metric on } X. \text{ Then } (X,G) \text{ is a } G\text{-Metric space.}
\]
Let \( f \) and \( g \) be two self maps on \( X \) defined by
\[
f(x) = \begin{cases} \frac{x}{2} & \text{if } x \leq 5 \\
2 - \frac{x}{5} & \text{if } x > 5 \\
\end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{x}{5} & \text{if } x \leq 5 \\
\frac{x}{2} & \text{if } x > 5 \\
\end{cases}
\]
Define \( \phi : [0,\infty) \to [0,\infty) \) by \( \phi(t) = \frac{2t}{3} \). Here \( f \) and \( g \) are \( f \)-compatible and \( f \)-weakly reciprocally continuous.

To see this, consider a sequence \( \{ x_n \} = \left\{ 5 - \frac{1}{n} \right\} \) for all \( n \). Then \( f x_n \to 5, g x_n \to 5 \) and \( G(f x_n, g x_n, g x_n) \to 0 \). Also \( f g x_n = 5 = f(5) \), \( g g x_n = 5 = g(5) \). Further, \( f \) and \( g \) satisfies all the conditions of Theorem 2.2 and have a unique common fixed point at \( x = 5 \).

**Corollary 2.1:** Let \( f \) and \( g \) be \( f \)-weakly reciprocally continuous self mappings on a G-complete G-Metric space \((X,G)\) with \( f X \subseteq g X \). Let \( \alpha \in (0,1) \) and
\[
\int_0^1 \varphi(t) dt \leq \alpha \left( \int_0^1 \varphi(t) dt \right)
\]
for all \( x,y,z \in X \), where \( \varphi \in \Psi \) and \( L(x,y,z) = \max \{ G(gx, fy, fz), G(gx, fy, fz), G(gx, fz, fy), G(gx, fz, fy), G(gx, fy, fz) \} \).

If \( f \) and \( g \) are either compatible or g-compatible or \( f \)-compatible or compatible of type (P), then \( f \) and \( g \) have a unique common fixed point.

**Proof:** Put \( \phi(t) = \alpha t \) where \( \alpha \in (0,1) \) in Theorem 2.2.

**Corollary 2.3:** Let \( f \) and \( g \) be \( f \)-weakly reciprocally continuous self mappings on a G-complete G-Metric space \((X,G)\) with \( f X \subseteq g X \). Let \( \alpha \in (0,1) \) and \( G(f x, f y, f z) \leq \alpha (L(x,y,z)) \) for all \( x,y,z \in X \), where \( L(x,y,z) = \max \{ G(gx, fy, fz), G(gx, fy, fz), G(gx, fy, fz), G(gx, fy, fz), G(gx, fy, fz) \} \).

If \( f \) and \( g \) are either compatible or g-compatible or \( f \)-compatible or compatible of type (P), then \( f \) and \( g \) have a unique common fixed point.

**Proof:** Put \( \varphi(t) = 1 \) and \( \phi(t) = \alpha t \) where \( \alpha \in (0,1) \) in Theorem 2.2.

The next result involves a cyclic integral contraction for a finite family of nonempty closed subsets of a G-Metric space \( X \).
Theorem 2.5: Let \((X, G)\) be a \(G\)-complete \(G\)-Metric space and \(\{A_i\}_{i=1}^m\) be a family of nonempty \(G\)-closed subsets of \(X\) with \(Y = \bigcup_{i=1}^m A_i\). Let \(T: Y \to Y\) be a map satisfying \(T(A_i) \subseteq A_{i+1}\) for all \(i=1,2,...,m\) where \(A_{m+1} = A_1\). Suppose that there exists \(\phi \in \Phi\) such that

\[
\int_0^\infty \varphi(t)\, dt \leq \phi\left( \int_0^\infty \varphi(t)\, dt \right)
\]

for all \(x, y, z \in A_i\) where \(\varphi \in \Psi\) and \(L(x, y, z)\) is defined as:

\[
L(x, y, z) = \max\left\{ \frac{G(x, y, z)}{G(x, Ty, Tz)} + \frac{G(y, Tx, Tz)}{G(y, Ty, Tz)} + \frac{G(z, Tx, Ty)}{G(z, Ty, Tx)} \right\}
\]

Then \(T\) has a unique fixed point in \(\bigcap_{i=1}^m A_i\).

Proof: Let \(x_0 \in A_i\) be any arbitrary point. Define a sequence \(\{x_n\}\) by \(x_n = T^n x_0, n = 1, 2, 3, ...\). Now we have \(x_0 \in A_1, x_1 \in A_2, ...\), since \(T\) is a cyclic mapping. If \(x_{n+1} = x_n\) for some \(n\), then \(x_n\) is the fixed point of \(T\). Therefore assume that \(x_{n+1} \neq x_n\) for all \(n\). Let \(\varepsilon > 0\).

As in Theorem 2.2 one can prove that

\[
\int_0^\infty \varphi(t)\, dt < \frac{1}{2} [\varepsilon - \varphi(\varepsilon)] \quad \forall n \geq 1
\]

Let \(k, n \in \mathbb{N} \cup \{0\}\) with \(k > n\). Then by following the lines in the proof of Theorem 2.2, we can prove that

\[
\int_0^\infty \varphi(t)\, dt < \varepsilon \quad \forall k \geq n \geq l
\]

Since \(\varepsilon\) is arbitrary, we have

\[
\int_0^\infty \varphi(t)\, dt \to 0 \quad \text{as } n, m \to \infty
\]

Therefore \(G(x_m, x_n, x_{n+1}) \to 0\) as \(n, m \to \infty\). That is \(\{x_n\}\) is a \(G\)-cauchy sequence in \(X\). Since \(X\) is \(G\)-complete there exists a point \(u\) in \(X\) such that \(\lim x_n = \lim T_{n-1} x_1 = u\).

Now we will prove that \(u \in \bigcap_{i=1}^m A_i\). Since \(x_0 \in A_1\), the subsequence \(\{x_{m(n-1)}\}_{n=1}^\infty \subseteq A_1\), \(\{x_{m(n-1)+1}\}_{n=1}^\infty \subseteq A_2\), ..., \(\{x_{m_{n-1}}\}_{n=1}^\infty \subseteq A_m\). All the \(m\) subsequences are \(G\)-convergent in the \(G\)-closed sets \(\{A_i\}_{i=1}^m\) and hence they all convergent to the same limit \(u \in \bigcap_{i=1}^m A_i\).

Next we prove that \(u = T u\). Consider

\[
\int_0^\infty \varphi(t)\, dt \leq \varphi\left( \int_0^\infty \varphi(t)\, dt \right)
\]

where

\[
L(u, u, u) = \max\left\{ \frac{G(u, u, u)}{G(u, Tu, Tu)} + \frac{G(u, Tu, Tu)}{G(u, Tu, Tu)} + \frac{G(u, Tu, Tu)}{G(u, Tu, Tu)} \right\}
\]

Letting \(n \to \infty\) in the above equation we obtain

\[
\int_0^\infty \varphi(t)\, dt \leq \varphi\left( \int_0^\infty \varphi(t)\, dt \right)
\]

where \(L(u, u, u) \leq G(u, Tu, Tu)\). Therefore
which implies $T_u = u$. Thus $u$ is a fixed point of $T$. The uniqueness of the fixed point can be proved easily.

We now provide an example to illustrate this theorem.

**Example:** Let $X = [-3, +3]$ and $A_i = \left[-\frac{3}{2}, \frac{3}{2}\right]$, $i = 1, 2, \ldots$ Let $Y = \bigcup_{i=1}^{\infty} A_i = [-2.2]$ and define $T: Y \rightarrow Y$ by

$$
T(y) = \frac{3y}{2}.
$$

Then $T(A_i) \subseteq A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$. Define the function $G: X \times X \times X \rightarrow [0, \infty)$ by $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Then $(X, G)$ is a $G$-Metric space.

Define $\phi: [0, \infty) \rightarrow [0, \infty)$ by $\phi(z) = \frac{z}{2}$.

Without loss of generality assume that $\frac{3}{2} \leq y \leq x$. Then,

$$
G(Tx, Ty, Tz) = G\left(\frac{-x}{2}, \frac{-y}{2}, \frac{-z}{2}\right) = x - z = \phi(2(x - z)) = \phi G(x, y, z)
$$

$\forall x, y, z \in X$. Therefore $T$ satisfies the contractive condition (8).

Further, $T$ satisfies all the hypotheses of Theorem 2.5 and have a unique fixed point at $x = 0 \in \bigcap_{i=1}^{\infty} A_i$.

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