Peristaltic Transport of a Herschel-Bulkley Fluid in a Non-Uniform Channel with Wall Effects

G.C. Sankad, P.S. Nagathan, Asha Patil, M.Y. Dhange

Abstract—The present study considers the peristaltic transport of a Herschel-Bulkley fluid in a two-dimensional channel of non-uniform cross section with wall properties. Using long wavelength approximation with low Reynolds number and dynamic boundary conditions, analytical expressions have been obtained stream function and time average velocity. The effects of pertinent parameters on these flow variables have been studied in both converging and diverging channel. It is observed that time average velocity increases with rigidity and stiffness in the wall.

Index Terms—Peristalsis, Herschel-Bulkley fluid, dynamic boundary conditions, power-law index, yield stress.

I. INTRODUCTION

Peristalsis pumping is a form of fluid transport that occurs when a progressive wave of area contraction or expansion propagates along the length of a distensible tube containing the fluid. Peristalsis appears to be the main mechanism for fluid transport in many physiological situations such as urine transport from the kidney to the bladder through the ureter, the movement of chyme in the gastrointestinal tract, the transport of spermatozoa in the ductus efferentes of the male reproductive tract and in the cervical canal, the movement of ova in the fallopian tubes, the transport of lymph in the lymphatic vessels, and blood flow in cardiac chambers. Also some biomedical instruments, such as heart lung machine use peristalsis to pump blood while mechanical devices like roller pumps use this mechanism to pump slurries and other corrosive fluids.

The study of peristalsis has received considerable attention in the last few decades mainly because of its relevance to engineering and biological systems. Several studies have been made analyzing both theoretical and experimental aspects of the peristaltic motion of Newtonian and non-Newtonian fluids in different situations (Fung and Yih [1], Shapiro et al. [2], Radhakrishnamacharya [3], Misery and Shehawey [4], Mishra and Rao [5], Srinvasacharya et al. [6], Hayat et al. [7], Kothandapani and Srinivas [8], Sobh [9]).

Though Newtonian and several non-Newtonian models have been used to study the motion of blood, it is realized that the Herschel-Bulkley model is a better model (Blair and Spanner 1974) and describes the behavior of blood very closely.

The Herschel-Bulkley fluid is a generalized model of a non-Newtonian fluid introduced by Hershel and Bulkley in 1926. Herschel-Bulkley fluids are materials possessing yield value but in flow they may exhibit the characteristics of shear thinning or shear thickening materials. They are the empirical combination of a Bingham fluid and power-law fluids. It is widely used in industry for the flow of pastes, slurries, polymer processing and other materials which exhibit a yield stress. Hence, the peristaltic transport of a Herschel-Bulkley fluid has received some attention in last few decades. Several attempts (Vajravelu et al. [10], Maruthi Prasad and Radhakrishnamacharya [11] and Medhavi [12]) have been made to understand the peristaltic motion of a Herschel-Bulkley fluid under various conditions.

In view of the importance of the interaction of peristalsis with the elastic properties of the wall, Mittra and Prasad [13] studied peristaltic transport of a Newtonian fluid in a two-dimensional uniform channel considering the elasticity of the walls. They solved this problem under the approximation of small amplitude ratio with dynamic boundary conditions. Muthu et al. [14] extended the analysis of Mittra and Prasad [13] to micropolar fluids.

It is known that many ducts in physiological and engineering systems are of non-uniform cross section. Keeping this in view, Gupta and Seshadri [18] studied peristaltic transport of Newtonian viscous fluid in a non-uniform duct without considering the elasticity of the wall. Mekheimer [19] considered peristaltic transport of Newtonian fluid through uniform and non-uniform annulus. Srinivasulu and Radhakrishnamacharya [17] investigated peristaltic
motion of a Newtonian fluid in a non-uniform channel with wall effects. Sankad and Radhakrishnamacharya [20] studied the effect of wall properties on peristaltic transport of micropolar fluid in a non-uniform channel. MHD flow of a Newtonian fluid through a porous medium in an asymmetric channel with peristalsis was investigated by Nagendra [21]. Also, Nagachadrakala et al. [22] considered slip conditions on MHD peristaltic flow of a hyperbolic tangent fluid in a channel with wall properties.

The importance of peristaltic transport in an asymmetric channel has been brought out by Eytan and Elad [23] with application to intra-uterine fluid flow in a non-pregnant uterus. In human body, several small blood vessels, lymphatic vessels, intestines, ductus efferentes of the reproductive tracts are generally observed to be non-uniform (Haynes [24], Srivastava et al.[ 25], Lee and Fung [26], Vatistas and Ghaly [27]). For example the vas deferens in rhesus monkey is in the form of a diverging tube with a ratio of exit to inlet dimensions of approximately 4 Guha et al. [28].

However, the effect of peristaltic pumping on flow of Herschel-Bulkley fluid in a non-uniform channel with wall effects has not been studied. Hence, an attempt has been made in this paper to understand the peristaltic flow of Herschel-Bulkley fluid in a non-uniform two dimensional channel with dynamic boundary conditions under long wave length approximation and the Reynolds to be low. Expressions for stream function and average velocity have been derived and the effects of various parameters on these flow variables have been investigated.

II. MATHEMATICAL FORMULATION

Consider the flow of Herschel-Bulkley fluid through a two dimensional channel of non-uniform cross section with flexible walls. It is assumed that progressive sinusoidal waves propagate along the length of the channel. A rectangular cartesian coordinate system \((x, y)\) is chosen with \(x\)-axis aligned with the central line of the channel and in the direction of propagation of waves.

The wall deformation due to the propagation of an infinite train of peristaltic waves is represented by

\[
y = \eta(x, t) = d + b'x + a\sin\frac{2\pi}{\lambda}(x - ct)
\]

where \(d\) is the half width at the inlet, \(b'\) is a constant whose magnitude is much less than unity, \(t\) is the time, \(a\) is the amplitude, \(\lambda\) is the wave length and \(c\) is the speed of the traveling waves (Fig. 1)

![Fig -1: Geometry of the problem](image)

The governing equation of motion of the flexible wall may be expressed as

\[
L(\eta) = p - p_0
\]

where \(L\) is the operator, which is used to represent the motion of the stretched membrane with damping forces such that

\[
L = -T \frac{\partial^2}{\partial x^2} + m \frac{\partial^2}{\partial t^2} + C \frac{\partial}{\partial t}
\]

Here, \(T\) is the elastic tension in the membrane, \(m\) is the mass per unit area and \(C\) is the coefficient of the viscous damping force and \(p_0\) is the pressure on the outside surface of the wall due to the tension in the muscles. We assume \(p_0 = 0\).
The equations governing the flow of Herschel-Bulkley fluid for the present problem, under long wavelength approximation and neglecting inertia terms are

\[
\frac{\partial}{\partial y}(\tau_{yx}) = -\frac{\partial p}{\partial x}
\]

(4)

where \( \tau_{yx} \) for Herschel-Bulkley fluid is given by

\[
\left(\frac{-\partial u}{\partial y}\right)^n = \frac{1}{\mu}(\tau_{yx} - \tau_0), \quad \tau_{yx} \geq \tau_0
\]

(5)

\[
\frac{\partial u}{\partial y} = 0, \quad \tau_{yx} \leq \tau_0
\]

(6)

where \( p \) is the pressure, \( \mu \) and \( n \) (\( n>1 \)) are consistency and flow behavior indices respectively.

Relation (6) corresponds to the vanishing of the velocity gradient in the region in which \( \tau_{yx} \leq \tau_0 \) and implies a plug flow. When shear stress in the fluid is very high (\( \tau_{yx} \geq \tau_0 \)), the power law behavior is indicated. It is noted that above Herschel-Bulkley fluid model reduces to Bingham plastic when \( n=1 \); power law fluid when \( \tau_0 = 0 \) and Newtonian fluid when \( n=1 \) and \( \tau_0 = 0 \). It is important to mention that the plug core width increases with yield stress \( \tau_0 \) and also with the flow behavior index \( n \).

The corresponding boundary conditions are

\[
u = 0 \quad \text{at} \quad y = \eta
\]

(7)

\[
\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad \tau_{yx} = 0 \quad \text{at} \quad y = 0
\]

(8)

The dynamic boundary conditions at the flexible walls, following Mittra and Prasad can be written as

\[
\frac{\partial L(\eta)}{\partial x} = \frac{\partial p}{\partial x} = -T\frac{\partial^3 \eta}{\partial t^3 \partial x} + m\frac{\partial^3 \eta}{\partial t^3 \partial x} + C\frac{\partial^2 \eta}{\partial t \partial x}
\]

(9)

Further, defining the stream function \( \psi \) by

\[
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}
\]

(10)

and introducing the following non-dimensional quantities

\[
p' = \frac{pd^{n+1}}{\lambda \mu c^n}, \quad \tau_0' = \frac{\tau_0}{\mu(c/d)^n}, \quad \tau_{yx}' = \frac{\tau_{yx}}{\mu(c/d)^n}, \quad b' = \frac{b \lambda}{d},
\]

\[
p' = \frac{pd^{n+1}}{\lambda \mu c^n}, \quad \tau_0' = \frac{\tau_0}{\mu(c/d)^n}, \quad \tau_{yx}' = \frac{\tau_{yx}}{\mu(c/d)^n}, \quad b' = \frac{b \lambda}{d},
\]

(11)

into equations (4-9), we get (after dropping primes)

\[
\frac{\partial}{\partial y}(\tau_{yx}) = -\frac{\partial p}{\partial x}, \quad \tau_{yx}' = \left(-\frac{\partial u}{\partial y}\right)' + \tau_0, \quad \tau_{yx} \geq \tau_0
\]

(12)

\[
\frac{\partial u}{\partial y} = 0, \quad \tau_{yx} \leq \tau_0
\]

(13)

where \( \tau_{yx} \) and \( \tau_0 \) are dimensionless shearing and yield stress respectively.
The boundary conditions are

\[ \psi = 0; \quad \psi_y = 0 \quad \text{and} \quad \tau_{yy} = 0 \quad \text{at} \quad y = 0, \]

(14)

\[ \frac{\partial p}{\partial x} = E_1 \frac{\partial^3 \eta}{\partial x^3} + E_2 \frac{\partial^3 \eta}{\partial t^2 \partial x} + E_3 \frac{\partial^2 \eta}{\partial t \partial x}, \quad \psi_y = 0 \quad \text{at} \quad y = \eta = 1 + bx + \varepsilon \sin 2\pi(x-t) \]

(15)

where \( \varepsilon = a/d \) is the amplitude ratio, \( E_1 = \frac{Td}{\lambda^3 \mu c^n} \), \( E_2 = \frac{m n^2}{\lambda^3 \mu c^{n-2}} \) and \( E_3 = \frac{C d}{\lambda^2 \mu c^{n-1}} \) are the non-dimensional elasticity parameters.

The non-dimensional quantities \( E_1, E_2 \) and \( E_3 \) are related to the wall motion through the dynamic boundary condition (15). The parameters \( E_1 \) and \( E_2 \) respectively represent the rigidity and stiffness of the wall. The viscous damping force in the wall is represented by \( E_3 \). In particular, \( E_3 = 0 \) implies that the walls move up and down with no damping force on them and hence indicates the case of elastic walls (i.e. \( E_3 = 0 \)).

Solving equation (12) subject to the boundary conditions (14) and (15), we obtain the expression for velocity as

\[ u_c = \frac{1}{P(k+1)} \left[ (P \eta - \tau_0)^{k+1} - (P \eta - \tau_0)^{k+1} \right]. \]

(16)

where \( P = -\frac{\partial p}{\partial x} \) and \( k = \frac{1}{n} \). We find the upper limit of the plug flow region by using the boundary condition that \( \psi_y = 0 \) at \( y = y_0 \) so that \( y_0 = \frac{\tau_0}{P} \). Also, by using the condition \( \tau_{yy} = \tau_{\eta} \) at \( y = \eta \), we obtain \( P = \frac{\tau_0}{\eta} \). Hence, \( \frac{y_0}{\eta} = \frac{\tau_0}{\tau_{\eta}}, \quad 0 < \tau < 1. \)

(17)

The expression for the fluid velocity in the plug flow region, \( u_p \), is obtained by substituting \( y = y_0 \) in equation (17) and this obviously satisfies equation (13) in the plug flow region.

Hence, we get

\[ u_p = \frac{1}{P(k+1)} \left[ (P \eta - \tau_0)^{k+1} \right]. \]

(18)

Integrating equations (16) and (18) and using the conditions \( \psi_p = 0 \) at \( y = 0 \) and \( \psi = \psi_p \) at \( y = y_0 \), we obtain the stream function as

\[ \psi = \frac{P^k}{k+1} \left[ \eta(y - y_0)^{k+1} - \frac{1}{k+2}(y - y_0)^{k+2} \right] \quad \text{for} \quad y_0 \leq y \leq \eta \]

(19)

\[ \psi_p = \frac{P^k}{k+1} \eta(y - y_0)^{k+1} \quad \text{for} \quad 0 \leq y \leq y_0 \]

(20)

Averaging equations (16) and (18) over one period of the motion yields the average velocity \( \bar{u} \) as

\[ \bar{u} = \frac{1}{T} \int_0^T u \, dt \]

(21)
III. RESULTS AND DISCUSSION

The effects of various parameters $E_1, E_2, E_3, n, \tau_0$, and $b$ on the time average velocity $\bar{u}$ have been computed numerically by using Mathematica software and the results are graphically depicted in figures (2-19).

The average velocity for the present problem depends upon the following important non-dimensional quantities.

- $E_1, E_2$, and $E_3$, the wall parameters which characterize the viscoelastic behavior of the flexible walls.
- The power-law index $n$ determines the non-linear behavior of the fluid. For $n<1$, it describes shear thinning and for $n>1$, shear thickening behavior.
- The yield stress $\tau_0$.

It can be observed from figures (2-7) that the time average velocity $\bar{u}$ increases with rigidity, stiffness and dissipative nature of the wall ($E_1, E_2$, and $E_3$) both in converging ($b<0$) and diverging channels ($b>0$).

The effect of power-law index $n$ on the time average velocity $\bar{u}$ in converging ($b<0$) and diverging ($b>0$) channels is shown in figures (8-11). It is observed that for converging as well as diverging channels, the time average velocity decreases with power-law index $n$ in the presence and absence of stiffness as well as dissipative effects in the wall.

![Figure 2: Effect of $E_1$ on average velocity $\bar{u}$](image1)

![Figure 3: Effect of $E_1$ on average velocity $\bar{u}$](image2)

![Figure 4: Effect of $E_2$ on average velocity $\bar{u}$](image3)

![Figure 5: Effect of $E_2$ on average velocity $\bar{u}$](image4)
Figures (12 and 13) show that for the case of shear thinning ($n < 1$), the time average velocity $\bar{u}$ decreases with yield stress and flow reversal takes place when there is no dissipative effects in the channel wall ($E_3 = 0$). Also, the time average velocity $\bar{u}$ decreases as yield stress $\tau_0$ increases when there is stiffness.
(E_2 \neq 0) and dissipative effects (E_3 \neq 0) in the wall for the case of shear thickening (n>1) [Figures 14 and 15]. This is true in both converging [Figures 12 and 14] and diverging [Figures 13 and 15] channels.

Figure 12: Effect of τo on average velocity \( \bar{u} \) 
(ε = 0.6, E_1 = 0.5, E_2 = 0.1, E_3 = 0.0, n = 0.2, b = −0.25, x = 0.2)

Figure 13: Effect of τo on average velocity \( \bar{u} \) 
(ε = 0.6, E_1 = 0.5, E_2 = 0.1, E_3 = 0.0, n = 0.2, b = 0.25, x = 0.2)

Figure 14: Effect of τo on average velocity \( \bar{u} \) 
(ε = 0.6, E_1 = 0.5, E_2 = 0.1, E_3 = 0.1, n = 1.2, b = −0.25, x = 0.2)

Figure 15: Effect of τo on average velocity \( \bar{u} \) 
(ε = 0.6, E_1 = 0.5, E_2 = 0.1, E_3 = 0.1, n = 1.2, b = 0.25, x = 0.2)

Figure 16: Effect of b on average velocity \( \bar{u} \) 
(ε = 0.6, E_1 = 5, E_2 = 0.0, E_3 = 0.0, n = 0.5, τo = 0.2, x = 0.2)

Figure 17: Effect of b on average velocity \( \bar{u} \) 
(ε = 0.6, E_1 = 5, E_2 = 0.4, E_3 = 0.5, n = 0.5, τo = 0.2, x = 0.2)

We can see from figures (16-19) that the time average velocity \( \bar{u} \) is more in a diverging channel compared to its value in a converging channel for the case of shear thinning (n<1) as well as for the case of shear thickening (n>1).
IV. CONCLUSIONS

In this study, the peristaltic transport of a Herschel-Bulkley fluid in a two-dimensional non-uniform channel with wall effects under dynamic boundary conditions has been analyzed. The governing equations have been linearised under long wave length approximation and analytical expressions for average velocity have been derived. The effects of various parameters on time average velocity have been studied through graphs. We conclude the following observations.

- It is observed that for both in converging and diverging channel, the time average velocity increases with power-law index.
- The time average velocity is high in diverging channel as compared to its value in converging channel.
- It is interesting to observe that for the case of shear thinning, flow reversal takes place when there is no viscous damping force in the converging channel.

REFERENCES


AUTHOR BIOGRAPHY

Dr.G.C.Sankad is working as Associate Professor, Mathematics Department in BLDEA’s VP Dr.PGH College of Engg. & Tech. Biajpur, Karnataka. He has done M.Sc. (Mathematics) & PGDCA from Karnatak University Dharwad (Karnataka) and Ph.D. in fluid Mechanics form NIT Warangal (A.P.) & He has a teaching experience of more than 20 years. He has several publications reputed international journals. He is also a life member of ISTAM & ISTE
Mrs. P. S. Nagathan is working as Assistant Professor, Mathematics Department in BLDEA’s VP Dr. PGH College of Engg. & Tech. Bijapur, Karnataka. She has done M.Sc., M.Phil. in Mathematics from Karnatak University Dharwad (Karnataka) and pursuing Ph.D. in fluid Mechanics & she has a teaching experience of 14 years. She is also a member of ISTE.

Mrs. Asha. B. Patil is working as Assistant Professor, Mathematics Department in BLDEA’s VP Dr. PGH College of Engg. & Tech. Bijapur, Karnataka. She has done M.Sc. (Mathematics), PGDCA. From Karnatak University Dharwad (Karnataka) and pursuing Ph.D. in fluid Mechanics & she has a teaching experience of 10 years. She is also a member of ISTE.

Mr. M.Y. Dhange is working as Assistant Professor, Mathematics Department in BLDEA’s VP Dr. PGH College of Engg. & Tech. Bijapur, Karnataka. He has done M.Sc. (Mathematics) from Shivaji. University Kolhapur, M.Phil. (Mathematics) from S.V. University Tirupati (A.P.) and pursuing Ph.D. in fluid Mechanics & He has a teaching experience of 10 years. He is also a member of ISTE.