On Special Boolean Like Rings

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Abstract: In this paper we introduce the concept of a special Boolean-like ring and prove that every Boolean like ring of A.L.Foster is a special Boolean-like ring. Examples are given to show that the converse is not true in general. We prove that every special Boolean-like ring is a subdirect product of a family of rings each of which is either a two element field or a four element Boolean like ring $H_4$ or a zero ring (i.e., product of any two elements is zero). In addition we prove that if $R$ is a special Boolean-like ring, then its Jacobson radical $J(R)$ and its nil radical $N$ are equal.

I. INTRODUCTION

Boolean rings have been generalized in a variety of ways by several mathematicians. Among these, Boolean-like rings of A.L.Foster[1] arise naturally from general ring-duality considerations and preserving many of the formal properties, both ring and ‘logical’, of Boolean rings. In this paper we introduce the concept of a special Boolean-like ring which is a generalization of Boolean-like ring. Further we furnish examples of special Boolean-like rings which are not Boolean-like rings. In section 1 we prove that if $R$ is a commutative ring with unity, then $R$ is a special Boolean-like ring iff it is a Boolean-like ring. In section 2, we prove that each special Boolean – like ring is a subdirect product of a family of rings, each of which is either a two element field or a four element Boolean – like ring $H_4$ or a zero-ring. In section 3 we prove that the Jacobson radical of a special Boolean – like ring $R$ is same as the nilradical of $R$.

II. SPECIAL BOOLEAN LIKE RINGS AND THEIR PROPERTIES

We start with a definition

Definition(1.1): A Commutative ring $R$ is called a special Boolean-like ring if (i) $r+r = 0$ for all $r \in R$, (ii) every element of $R$ can be expressed as the sum of an idempotent and a nilpotent element of $R$, and (iii) $mn = 0$ for nilpotent elements $m, n$ in $R$.

Note (1.2) [7]: (i) Every homomorphic image of a special Boolean-like ring is also a special Boolean-like ring.

(ii) If a special Boolean-like ring $R$ has no nonzero nilpotent elements, then by (ii) of the above definition, $R$ is a Boolean ring.

(iii) If $a$ is a nilpotent element, then by taking $m = n = a$ in (iii) of the above definition, $aa = 0$, i.e., $a^2 = 0$.

According to A.L.Foster[5], a commutative ring $R$ with unity is a Boolean-like ring if it is of characteristic 2 and $x(1+x)y(1+y) = 0$ for all $x, y \in R$.

We show that every Boolean like ring is a special Boolean-like ring.

Lemma (1.3): If $R$ is a Boolean-like ring, then

(i) $a^4 = a^2$ (i.e. $a^2$ is an idempotent) for all $a \in R$

(ii) an element $r \in R$ is nilpotent if and only if $r^2 = 0$
Proof: (i) If $a \in R$, then by taking $x=y=a$ in the definition we get that $a(1+a)(1+a) = 0$

By expanding, we have that $(a+a^2)(a+a^2) = 0 \Rightarrow a^2 + a^3 + a^4 = 0 \Rightarrow a^2 + a^4 = 0 \Rightarrow a^2 = a^4$

since characteristic of $R$ is 2.

(ii) If $r$ is nilpotent, then $r^m = 0$ for some $m>1$. W.l.o.g we may assume that $m$ is even i.e., $m = 2n$ for some $n$. Therefore $0 = a^n = a^{2n} = (a^2)^n = a^2$, since $a^2$ is an idempotent by (i). Converse is trivial.

Theorem (1.4): Every Boolean-like ring is a special Boolean-like ring.

Proof: Let $R$ be Boolean-like ring. We first show that every element of $R$ is a sum of an idempotent element and a nilpotent element. If $x \in R$, then $x = x^2 + x + x^2$, since characteristic of $R$ is 2. Since $x^2$ is an idempotent, by the above lemma, it suffices if we show that $x + x^2$ is nilpotent. Consider $(x + x^2)^2 = [x(1+x)]^2 = 0$, since $R$ is a Boolean-like ring. Hence $x + x^2$ is nilpotent. If $r, s$ are nilpotent elements, then $r^2 = s^2 = 0$, by the above lemma. Therefore $rs = (r+r^2)(s+s^2) = r(1+r)s(1+s) = 0$. Hence $R$ is a special Boolean-like ring.

Example (1.5): Let $R$ be a commutative ring with characteristic 2 and let $J$ be the set of all idempotents in $R$. Then $J$ is a subring of $R$. The set $J \times R$ is a special Boolean-like ring under the addition and multiplication defined as follows:

$(a_1, r_1) + (a_2, r_2) = (a_1+a_2, r_1+r_2)$

$(a_1, r_1) (a_2, r_2) = ((a_1a_2, a_1r_2+a_2r_1)$ for all $(a_1, r_1), (a_2, r_2) \in J \times R.$

Note: (i) Note that $(a, r) \in J \times R$ is an idempotent

$\iff (a, r) (a, r) = (a, r) \iff (a^2, ar+ar) = (a,r) \iff (a,0) = (a,r) \iff r = 0$. Thus $(a,r)$ is an idempotent iff $r = 0$.

(ii) Also $(a,r)$ is nilpotent $\Rightarrow (a,r)^m = (0,0)$ for some $m > 1$. W.l.o.g we may assume that $m$ is even. Then $m = 2n$. Therefore, $(a,r)^{2n} = (0,0) \Rightarrow [(a,r)^2]^n = (0,0) \Rightarrow (a,0)^n = (0,0) \Rightarrow (a,0) = (0,0) \Rightarrow a=0$

since $(a,0)$ is an idempotent by (i).

Conversely if $a = 0$, then $(a,r)^2 = (0,0) \Rightarrow (a,r)$ is nilpotent.

Therefore $(a,r)$ is nilpotent $\iff a = 0$.

As a particular case we get the following

Example (1.6): Let $Z_2 = \{0, 1\}$ be the ring of integers modulo 2. Then $Z_2 \times Z_2$ is a special Boolean – like ring with identity. Note that $Z_2 \times Z_2 = \{(0,0), (1,0), (0,1), (1,1)\}$.

Write $0 = (0,0), 1 = (1,0), p = (0,1), q = (1,1)$.

Addition and multiplication tables are as follows:
We denote this ring by $H_4$.

But in general the converse of theorem 1.4 is not true. For this we given a example.

**Example (1.7):** Let $R = \{0,a,b,c\}$ be the klein’s four group.

Addition '+' and multiplication '.' tables are given as follows:

\[
\begin{array}{cccc}
+ & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a \\
b & 0 & 0 & 0 & b \\
c & 0 & a & 0 & a \\
\end{array}
\]

\[
\begin{array}{cccc}
. & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array}
\]

R is a special Boolean-like ring but R is not a Boolean like ring.

**Theorem (1.8):** Let R be a commutative ring with unity 1. Then R is a special Boolean-like ring iff R is a Boolean-like ring.

**Proof:** Suppose R is a special Boolean-like ring. To show R is a Boolean-like ring, it suffices to show that $a(1+a)b(1+b) = 0$ for all $a,b \in R$. By (ii) of the definition [1.1], $a = e+m$ and $b = f+n$, where e, f are idempotents and m, n are nilpotent elements of R.

Now, $a(1+a) = (e+m)(1+e+m) = e + e^2 + em + m + me + m^2 = m$, since $m^2 = 0$ by note[1.2](iii). Similarly, we can prove that $b(1+b) = n$. Therefore, $a(1+a)b(1+b) = mn = 0$. Hence R is a Boolean-like ring. Converse follows from theorem 1.4.

**III. SUBDIRECT PRODUCT REPRESENTATION OF SPECIAL BOOLEAN LIKE RINGS**

In this section we prove that each special Boolean – like ring is a subdirect product of a family of rings, each of which is either a two element field or a four element Boolean – like ring $H_4$ or a zero-ring.
Lemma (2.1): If $R$ is a subdirectly irreducible Boolean ring then $R$ is a two element field.

**Proof:** Let $0 \neq a \in R$. Then $aR \cap a^* = 0$, where $a^* = \{ r \in R / ar = 0 \}$, since $t \in aR \cap a^* \Rightarrow t = ar$, where $r \in R$ and $at = 0 \Rightarrow 0 = at = aar = ar = t$. Therefore $a^* = (0)$, since $aR \neq (0)$. Let $v \neq 0$ be an arbitrary but a fixed element in $R$. Then by the above result, $v^* = (0)$. For any $r \in R$, $v(r+vr) = 0$. Since $v^* = (0)$, $r+vr = 0 \Rightarrow vr = r$. Thus $v$ is the identity element of $R$, which we denote by $1$. For $0 \neq s \in R$, $s(1+s) = 0 \Rightarrow 1 + s = 0 \Rightarrow s = 1$. Therefore, $R = \{0,1\}$ is a two element field.

Lemma (2.2): Suppose $R$ is a subdirectly irreducible special Boolean-like ring. If $R$ is not a Boolean ring and if $R$ contains a nonzero idempotent element, then $R \cong H_4$.

**Proof:** Let $J = \{ e \in R / e$ is an idempotent $\}$ and let $N$ be the nil radical of $R$. $J$ is a subring of $R$ and $N$ is an ideal of $R$. Since $R$ is not a Boolean ring, $R$ contains a nonzero nilpotent element, say $p$. If $0 \neq j \in J$, then $jR \cap j^* = (0)$ where $j^*$ is as in lemma 2.1. Since $jR \neq (0)$, $j^* = (0)$. Let $v \neq 0$ be an arbitrary but a fixed element in $J$. Then by the above, $v^* = (0)$. For $r \in R$, $v(r+vr) = 0$. Since $v^* = (0)$, $r+vr = 0 \Rightarrow vr = r$. Thus $v$ is the identity element of $R$, which we denote by $1$. If $0 \neq a \in J$, then $a(1+a) = 0$ and hence $1+a = 0 \Rightarrow a = 1$. Therefore $J = \{0,1\}$, where $1$ is the identity element of $R$. Let $\langle p \rangle = \{ rp / r \in R \}$. If $r \in R$, then define $e = e+n$, where $e \in J$ and $n \in N$. Therefore $pr = p(e+n) = pe + pn = pe$, by (iii) of the definition. Since $J = \{0,1\}$, we have that either $e = 0$ or $e = 1$. Hence $pe = 0$ or $pe = p$. Thus $pr \in \{0,p\}$. Thus $\langle p \rangle = \{0,p\}$. If $0 \neq q \in N$, then similarly $\langle q \rangle = \{0,q\}$. Since $R$ is irreducible $\langle p \rangle \cap \langle q \rangle \neq (0) \Rightarrow p = q$. Therefore $N = \{0,p\}$.

By (ii) of the def 1.1, $R = J + N = \{0,1\} + \{0,p\} = \{0,1,p,q\}$, where $q = 1+p$. So, $R \cong H_4$.

Where $H_4$ is as in example 1.6.

Lemma (2.3): Let $R$ be a subdirectly irreducible special Boolean-like ring. If $R$ does not contain nonzero idempotent elements, then $R$ is a zero ring.

**Proof:** By (ii) of the def 1.1, each element in $R$ is a nilpotent element. If $a, b \in R$, then $a, b$ are nilpotent elements. By (iii) of the def 1.1, $ab = 0$. Thus $ab = 0$ for $a, b \in R$.

Therefore $R$ is a zero ring.

Theorem (2.4): Every special Boolean-like ring is a subdirect product of special Boolean like rings $\{R_i\}$, where each $R_i$ is either a two element field or the four element Boolean like ring $H_4$, or a zero ring.

**Proof:** Let $R$ be a special Boolean like ring. By Birkhoff’s theorem, $R$ is a subdirect product of a family of rings $\{R_i\}$, where each $R_i$ is subdirectly irreducible. Since each $R_i$ is a homomorphic image of $R$, by note 1.2 (i), $R_i$ is a special Boolean like ring. By lemmas 2.1, 2.2, and 2.3, $R_i$ is a two element field or $R_i$ is isomorphic to $H_4$ or $R_i$ is a zero ring.
In this section we prove that if R is a special Boolean-like ring, then the Jacobson radical J ( R ) of R is equal to the nil radical N of R. In addition we prove several interesting results. We present below the definition of the Jacobson radical of a commutative ring [4].

**Definition 3.1:** (i) Let R be a commutative ring. An element $a \in R$ is said to be quasi-regular if there exists $r \in R$ such that $r + a + ar = 0$. The element r is called a quasi-inverse of a. An ideal I of R is said to be quasi-regular if every element of I is quasi-regular.

Note that if R has an identity 1, then $a \in R$ is quasi-regular if and only if $1 + a$ is invertible.

If R is a commutative ring, then there is an ideal J ( R ) of R such that J ( R ) is a quasi-regular ideal which contains every quasi-regular ideal of R. The ideal J ( R ) is called the Jacobson radical of the ring R. Before proving the main theorem we prove some interesting results.

**Lemma 3.2:** If R is a commutative ring, then every nil ideal is contained in the radical J ( R )

**Proof:** If $a^2 = 0$, let $r = -a + a^2 + a^3 + \ldots + (-1)^{n-1} a^{n-1}$. Then $r + a + ra = 0$ and hence a is quasi-regular and hence is contained in J ( R ).

**Lemma 3.3:** The set I of all idempotent elements in a special Boolean-like ring R forms a sub ring of R.

**Proof:** Since $0 \in I$, $1 \neq \varphi$. If $e, f \in I$, then $(e + f)^2 = e^2 + 2ef + f^2 = e + f$, since the characteristic of R is 2. Thus $e + f \in I$. Further $(ef)^2 = e^2f^2 = ef$ and this implies that ef $\in I$. Hence I is a subring of R.

**Lemma 3.4:** The set N of all nilpotent elements in a commutative ring R forms an ideal of R.

**Proof:** Trivially $N \neq \varphi$. If $a, b \in N$, then $a^{n_1} = b^{n_2} = 0$ for some integers $n_1 > 0, n_2 > 0$. Then

$$(a + b)^{n_1 + n_2} = \sum_{k=0}^{n_1 + n_2} \binom{n_1 + n_2}{k} a^k b^{(n_1 + n_2) - k} = 0$$

since $a^2 b^{n_2} - k = 0$ for all $k \geq 0$.

So, $a + b \in N$. Also $(-b)^{n_2} = (-1)^{n_2} (b)^{n_2} = 0$ and this implies that $-b \in N$. For any $r \in R$, $(ar)^{n_1} = a^{n_1} r^{n_2} = 0$ and this implies that $ar \in N$. Thus N is an ideal of R.

**Lemma 3.5:** If R is a special Boolean-like ring, then J(R) contains no nonzero idempotents.

**Proof:** Let e be an idempotent in R. If $e \in J(R)$ then e is quasi-regular. Hence there is an element $r \in R$ such that $r + e + re = 0 \Rightarrow e(r + e + re) = e0 = 0 \Rightarrow er + e + re = 0 \Rightarrow e = 0$.

Therefore J(R) contains no nonzero idempotents.

**Definition 3.6:** If R is a commutative ring, then the set N of all nilpotent elements in R is an ideal. The ideal N is called the nil radical of R.

We now prove the main theorem.

**Theorem 3.7:** If R is a special Boolean-like ring, then $J(R) = N$ where J ( R ) and N are the Jacobson radical and the nil radical of R respectively.

**Proof:** By definition 1.1, $R = N + I$. By lemma 3.2, $N \subseteq J(R)$. By modular law, $J(R) = J(R) \cap (N + I) = N + (J(R) \cap I) = N + (0) = N$, since $J(R) \cap I = (0)$ by lemma 3.5. Therefore $J(R) = N$.

Let us recall that a ring R is said to be (Jacobson) semi simple if its Jacobson radical $J(R) = (0)$.

**Corollary 3.8:** Every Boolean ring is semi simple.

**Proof:** Let R be a Boolean ring. Then it is a special Boolean-like ring. By theorem (3.7), $J(R) = N$, where N is the nil radical of R. Since R is a Boolean ring, $N = (0)$ and hence $J(R) = (0)$. Thus R is semi prime.
REFERENCES


