INTUITIONISTIC FUZZY
MULTISETS

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Abstract– In this paper a new concept named Intuitionistic Fuzzy Multiset is introduced. The basic operations on Intuitionistic Fuzzy Multisets such as union, intersection, addition, multiplication etc. are discussed. Cartesian product and αβ-cut of Intuitionistic Fuzzy Multisets are defined and their various properties are discussed. An efficient algorithm is developed for finding the αβ-cut and strong αβ-cut of Intuitionistic Fuzzy Multiset. Two new operators over the set of all Intuitionistic Fuzzy Multisets, which will transform every Intuitionistic Fuzzy Multisets into a Fuzzy Multiset are defined and their various properties are discussed.

Keywords : Intuitionistic Fuzzy set, Intuitionistic Fuzzy Multi-dimensional set, Multiset, Cartesian product, αβ-cut.

1 Introduction

The theory of sets, one of the most powerful tools in modern mathematics is usually considered to have begun with Georg Cantor (1845-1918). Considering

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the uncertainty factor, Lofti Zadeh[1] introduced Fuzzy sets in 1965, in which a membership function assigns to each element of the universe of discourse, a number from the unit interval to indicate the degree of belongingness to the set under consideration. After this, many new approaches and theories treating imprecision and uncertainty have been proposed. In 1983, Krassimir T. Atanassov[2],[3] introduced the concept of Intuitionistic Fuzzy sets (IFS). The same time a theory called ‘Intuitionistic Fuzzy set theory’ was independently introduced by Takeuti and Titani[4] as a theory developed in (a kind of) Intuitionistic logic. An Intuitionistic Fuzzy set is characterized by two functions expressing the degree of membership (belongingness) and the degree of nonmembership (nonbelongingness) of elements of the universe to the IFS.

Among the various notions of higher-order Fuzzy sets, Intuitionistic Fuzzy sets proposed by Atanassov provide a flexible framework to explain uncertainty and vagueness. IFS reflect better the aspects of human behavior. A human being who expresses the degree of belongingness of a given element to a set, does not often expresses the corresponding degree of non-belongingness as the complement. This psychological fact states that linguistic negation does not always coincides with logical negation. It is well-known that in the beginning of the last century L.Brouwer introduced the concept of Intuitionism. The name Intuitionistic Fuzzy set is due to George Gargove, with the motivation that their fuzzification denies the law of excluded middle—one of the main ideas of Intuitionism. This idea, which is a natural generalization of a standard Fuzzy set, seems to be useful in modelling many real life situations, like negotiation processes, psychological investigations, reasoning etc. The relation between Intuitionistic Fuzzy sets and other theories modelling imprecision can be seen in [5].

Many fields of modern mathematics have been emerged by violating a basic principle of a given theory only because useful structures could be defined this
way. While sets permit us to have atmost one occurrence of each element, multisets or bags, permit us to have multiple occurrences of the elements. Richard Dedekind is the first person who used the term multiset in his paper "Was sind und was sollen die Zahlen"("The nature and meaning of numbers")[6]. This paper was published in 1888. A complete account of the development of multiset theory can be seen in[7]. As a generalization of multiset, Yager[8] introduced the concept of Fuzzy Multiset(FMS). An element of a Fuzzy Multiset can occur more than once with possibly the same or different membership values.

This paper is an attempt to combine the two concepts: Intuitionistic Fuzzy sets and Fuzzy Multisets together by introducing a new concept called Intuitionistic Fuzzy Multisets.

2 Motivation and Application

Information retrieval (IR), in general, is the science of searching for documents in a database. Because the datas are clustered, there will be an overlapping in these procedures. An information retrieval process begins when a user enters a query into the system. Queries are nothing but formal statements of information needs. In information retrieval a query can be identified by several objects with different degrees of freedom. Depending on the application, the data objects may be, text documents, pictures, audios, videos etc. Nowadays major search engines like Google,Yahoo etc. are proposing services for mobile internet searching. So the need for a better model to handle different types of queries are essential and lot of research is going in these directions. The Fuzzy set theory offers a general framework for dealing with flexible queries. Especially the FMS plays an important role here[9],[10]. One of the limitations is, here we are dealing with precise data. But a typical user often formulates his requirements in a natural
language which contains imprecise expressions. So it is essential to use queries which allow for a more intelligent and human consistent information retrieval. Flexible queries are able to take users preferences and degrees of importance into account. Thus we can use the concept of IFS, as it represents the aspects of human behaviour. So there came the necessity of combining these two concepts by introducing IFMS. In this way the new structures are really "intuitionistic" as their name suggests. Although it seems to be a natural generalisation, IFMS gives an insight for using the flexible querying more effectively and efficiently.

Section(3) will give all basic definitions and notions for further study. In section(4) the concept of Intuitionistic Fuzzy Multisets have been introduced and discussed their basic operations. In section(5) Cartesian product, alpha-beta cut and two new operators are defined on IFMS and discussed their various properties. In section(6) a short comparison between the new concepts Intuitionistic Fuzzy Multisets and Intuitionistic Fuzzy Multi-dimensional sets are discussed.

In the next section a brief review of Fuzzy Multisets and Intuitionistic Fuzzy sets are given. Throughout the paper we assume that the underlying set is finite and counts of each element of a multiset is also finite.

3 Preliminaries

Definition 3.1. [1] Let $X$ be a nonempty set. A Fuzzy Set $A$ drawn from $X$ is defined as

$$A = \{<x, \mu_A(x)> : x \in X\}$$

where $\mu_A(x) : X \mapsto [0, 1]$ is the membership function of the Fuzzy Set $A$.

Definition 3.2. [8] Let $X$ be a nonempty set. A Fuzzy Multiset (FMS) $A$ drawn from $X$ is characterised by a function, ‘count membership’ of $A$ denoted by $CM_A$ such that $CM_A : X \mapsto Q$ where $Q$ is the set of all crisp multisets drawn
from the unit interval \([0,1]\). Then for any \(x \in X\), the value \(CM_A(x)\) is a crisp multiset drawn from \([0,1]\). For each \(x \in X\), the membership sequence is defined as the decreasingly ordered sequence of elements in \(CM_A(x)\). It is denoted by \((\mu_1^A(x), \mu_2^A(x), \ldots, \mu_p^A(x))\) where \(\mu_1^A(x) \geq \mu_2^A(x) \geq \ldots \geq \mu_p^A(x)\).

A complete account of the applications of Fuzzy Multisets in various fields can be seen in [11].

**Definition 3.3.** [3] Let \(X\) be a nonempty set. An Intuitionistic Fuzzy Set (IFS) \(A\) is an object having the form \(A = \{ < x : \mu_A(x), \nu_A(x) > : x \in X \}\), where the functions \(\mu_A : X \rightarrow [0, 1]\) and \(\nu_A : X \rightarrow [0, 1]\) define respectively the degree of membership and the degree of nonmembership of the element \(x \in X\) to the set \(A\) with \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\) for each \(x \in X\).

**Remark 3.4.** Every Fuzzy set \(A\) on a nonempty set \(X\) is obviously an IFS having the form

\[ A = \{ < x : \mu_A(x), 1 - \mu_A(x) > : x \in X \}\]

**Definition 3.5.** [3] \(\alpha\beta\)-cut of an IFS \(A\) is defined as

\[ \{ x : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta : 0 \leq \alpha + \beta \leq 1, x \in X \}\]

and strong \(\alpha\beta\)-cut as

\[ \{ x : \mu_A(x) > \alpha \text{ and } \nu_A(x) < \beta : 0 \leq \alpha + \beta \leq 1, x \in X \}\]

where \(\alpha, \beta \in [0, 1]\).

Using the definition of FMS and IFS, a new generalised concept can be defined as follows.

**4 Intuitionistic Fuzzy Multiset**

**Definition 4.1.** [12] Let \(X\) be a nonempty set. An Intuitionistic Fuzzy Multiset \(A\) denoted by IFMS drawn from \(X\) is characterised by two functions :
‘Count membership’ of $A$ ($CM_A$) and ‘count nonmembership’ of $A$ ($CN_A$) given respectively by $CM_A : X \rightarrow Q$ and $CN_A : X \rightarrow Q$ where $Q$ is the set of all crisp multisets drawn from the unit interval $[0,1]$, such that for each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in $CM_A(x)$ which is denoted by $(\mu_1^A(x), \mu_2^A(x), \ldots, \mu_p^A(x))$ where $\mu_1^A(x) \geq \mu_2^A(x) \geq \ldots \geq \mu_p^A(x)$ and the corresponding nonmembership sequence will be denoted by $(\nu_1^A(x), \nu_2^A(x), \ldots, \nu_p^A(x))$ such that $0 \leq \mu_i^A(x) + \nu_i^A(x) \leq 1$ for every $x \in X$ and $i = 1, 2, \ldots, p$.

An IFMS is denoted by
\[ A = \{ < x : (\mu_1^A(x), \mu_2^A(x), \ldots, \mu_p^A(x)), (\nu_1^A(x), \nu_2^A(x), \ldots, \nu_p^A(x) > : x \in X \} \]

**Remark 4.2.** Note that since we arrange the membership sequence in decreasing order, the corresponding nonmembership sequence may not be in decreasing or increasing order.

**Definition 4.3.** [12] Length of an element $x$ in an IFMS $A$ is defined as the Cardinality of $CM_A(x)$ or $CN_A(x)$ for which $0 \leq \mu_i^A(x) + \nu_i^A(x) \leq 1$ and it is denoted by $L(x : A)$. That is
\[ L(x : A) = |CM_A(x)| = |CN_A(x)| \]

**Definition 4.4.** [12] If $A$ and $B$ are IFMSs drawn from $X$ then
\[ L(x : A, B) = Max\{L(x : A), L(x : B)\} \]

We can use the notation $L(x)$ for $L(x : A, B)$. 
Example 4.5. [12] Consider

$X = \{x, y, z, w\}$

$A = \{< x : (0.3, 0.2), (0.4, 0.5) > , < y : (1, 0.5, 0.5), (0, 0.5, 0.2) > ,$
$< z : (0.5, 0.4, 0.3, 0.2), (0.4, 0.6, 0.6, 0.7) > \}$

$B = \{< x : (0.4), (0.2) > , < y : (1, 0.3, 0.2), (0, 0.4, 0.5) > ,$
$< w : (0.2, 0.1), (0.7, 0.8) > \}$

Here $L(x : A) = 2, L(y : A) = 3, L(z : A) = 4, L(w : A) = 0$

$L(x : B) = 1, L(y : B) = 3, L(z : B) = 0, L(w : B) = 2$

$L(x : A, B) = 2, L(y : A, B) = 3, L(z : A, B) = 4, L(w : A, B) = 2$

Next the basic operations on IFMS are defined. Note that we can make $L(x : A) = L(x : B)$ by appending sufficient number of 0’s and 1’s with the membership and nonmembership values respectively.

Definition 4.6. [12] For any two IFMSs $A$ and $B$ drawn from a set $X$ the following operations and relations will hold. Let

$A = \{< x : (\mu^1_A(x), \mu^2_A(x), ..., \mu^p_A(x)), (\nu^1_A(x), \nu^2_A(x), ..., \nu^p_A(x)) > : x \in X \}$ and

$B = \{< x : (\mu^1_B(x), \mu^2_B(x), ..., \mu^p_B(x)), (\nu^1_B(x), \nu^2_B(x), ..., \nu^p_B(x)) > : x \in X \}$ then

1. Inclusion

$A \subset B \iff \mu^j_A(x) \leq \mu^j_B(x)$ and $\nu^j_A(x) \geq \nu^j_B(x); j = 1, 2, ..., L(x), x \in X$

$A = B \iff A \subset B$ and $B \subset A$

2. Complement

$\neg A = \{< x : (\nu^1_A(x), ..., \nu^p_A(x)), (\mu^1_A(x), ..., \mu^p_A(x)) > : x \in X \}$.

3. Union$(A \cup B)$

In $A \cup B$ the membership and nonmembership values are obtained as follows.
\[
\mu^j_{A \cup B}(x) = \mu^j_A(x) \lor \mu^j_B(x) \\
\nu^j_{A \cup B}(x) = \nu^j_A(x) \land \nu^j_B(x) \\
j = 1, 2, \ldots, L(x), \ x \in X.
\]

4. Intersection \((A \cap B)\)

In \(A \cap B\) the membership and nonmembership values are obtained as follows.
\[
\mu^j_{A \cap B}(x) = \mu^j_A(x) \land \mu^j_B(x) \\
\nu^j_{A \cap B}(x) = \nu^j_A(x) \lor \nu^j_B(x) \\
j = 1, 2, \ldots, L(x), \ x \in X.
\]

5. Addition \((A \oplus B)\)

In \(A \oplus B\) the membership and nonmembership values are obtained as follows.
\[
\mu^j_{A \oplus B}(x) = \mu^j_A(x) + \mu^j_B(x) - \mu^j_A(x) \cdot \mu^j_B(x) \\
\nu^j_{A \oplus B}(x) = \nu^j_A(x) + \nu^j_B(x) - \nu^j_A(x) \cdot \nu^j_B(x) \\
j = 1, 2, \ldots, L(x), \ x \in X.
\]

6. Multiplication \((A \otimes B)\)

In \(A \otimes B\) the membership and nonmembership values are obtained as follows.
\[
\mu^j_{A \otimes B}(x) = \mu^j_A(x) \cdot \mu^j_B(x) \\
\nu^j_{A \otimes B}(x) = \nu^j_A(x) + \nu^j_B(x) - \nu^j_A(x) \cdot \nu^j_B(x) \\
j = 1, 2, \ldots, L(x), \ x \in X.
\]

Here \(\lor, \land, +, -\) denotes maximum, minimum, multiplication, addition, subtraction of real numbers respectively.

In (1)-(4) the verification of compatibility of operations is very easy. Regarding (5) and (6), the following two results are sufficient to reach the correctness.

**Proposition 1.** [3] For any real numbers \(a, b, c, d \in [0, 1]\) with \(0 \leq a + b \leq 1, 0 \leq c + d \leq 1\) we have \(0 \leq a + c - a.c \leq 1\) and \(0 \leq b + d - b.d \leq 1\).
Proposition 2. For any real numbers $a, b, c, d \in [0, 1]$ with $a \geq b, c \geq d$ we have $a + c - a.c \geq b + d - b.d$ and $a.c \geq b.d$

From the above two results it follows that $A \oplus B$ and $A \otimes B$ are IFMSs. The above defined operations can be illustrated with the IFMSs $A$ and $B$ given in Example 4.5. For comparison $A$ and $B$ can be written as follows.

Example 4.7.

$A = \{< x : (0.3, 0.2), (0.4, 0.5) >, < y : (1, 0.5, 0.5), (0, 0.5, 0.2) >, \}$
$B = \{< x : (0.4, 0), (0.2, 1) >, < y : (1, 0.3, 0.2), (0, 0.4, 0.5) >, \}$

Then

$A \cup B = \{< x : (0.4, 0.2), (0.2, 0.5) >, < y : (1, 0.5, 0.5), (0, 0.4, 0.2) >, \}$
$A \cap B = \{< x : (0.3, 0), (0.4, 1) >, < y : (1, 0.3, 0.2), (0, 0.5, 0.5) >, \}$

The above operations have the following properties.

Theorem 4.8. For any three IFMSs $A, B, C$

1. Commutative Law

$A \cup B = B \cup A$
$A \cap B = B \cap A$
2. **Idempotent Law**

\[ A \cup A = A \]
\[ A \cap A = A \]

3. **De Morgan’s Laws**

\[ \neg (A \cup B) = (\neg A \cap \neg B) \]
\[ \neg (A \cap B) = (\neg A \cup \neg B) \]

4. **Associative Law**

\[ A \cup (B \cup C) = (A \cup B) \cup C \]
\[ A \cap (B \cap C) = (A \cap B) \cap C \]

5. **Distributive Law**

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

**Proof.** For (1),(2) and (3) the proof is obvious. We prove (4) and (5).

Let \( A, B, \) and \( C \) be IFMSs defined as follows.

\[
A = \{ x \in X : (\mu_A^1(x), \mu_A^2(x), ..., \mu_A^p(x)), (\nu_A^1(x), \nu_A^2(x), ..., \nu_A^p(x)) > : x \in X \}
\]
\[
B = \{ x \in X : (\mu_B^1(x), \mu_B^2(x), ..., \mu_B^p(x)), (\nu_B^1(x), \nu_B^2(x), ..., \nu_B^p(x)) > : x \in X \}
\]
\[
C = \{ x \in X : (\mu_C^1(x), \mu_C^2(x), ..., \mu_C^p(x)), (\nu_C^1(x), \nu_C^2(x), ..., \nu_C^p(x)) > : x \in X \}
\]
The next theorem shows the relations connecting $\cup$.

To prove (4)

$A \cup (B \cup C) = \{ < x : (\mu_A(x), \ldots, \mu_A(x)), (\nu_A(x), \ldots, \nu_A(x)) > : x \in X \} \cup \{ < x : (\mu_B(x) \lor \mu_C(x), \ldots, \mu_B(x) \lor \mu_C(x)), (\nu_B(x) \land \nu_C(x), \ldots, \nu_B(x) \land \nu_C(x)) > : x \in X \} = \{ < x : ((\mu_A(x) \lor \mu_B(x)) \lor \mu_C(x), \ldots, (\mu_A(x) \lor \mu_B(x)) \lor \mu_C(x)), ((\nu_A(x) \lor \nu_B(x)) \lor \nu_C(x), \ldots, (\nu_A(x) \lor \nu_B(x)) \lor \nu_C(x)) > : x \in X \} = (A \cup B) \cup C$

Similarly we can prove $A \cap (B \cap C) = (A \cap B) \cap C$

To prove (5)

$A \cup (B \cap C) = \{ < x : (\mu_A(x) \lor (\mu_B(x) \land \mu_C(x)), \ldots, \mu_A(x) \lor (\mu_B(x) \land \mu_C(x)), (\nu_A(x) \land (\nu_B(x) \lor \nu_C(x), \ldots, (\nu_A(x) \land (\nu_B(x) \lor \nu_C(x)) > : x \in X \} \cup \{ < x : ((\mu_A(x) \lor \mu_B(x)) \land \mu_C(x), \ldots, (\mu_A(x) \lor \mu_B(x)) \land \mu_C(x)), ((\nu_A(x) \lor \nu_B(x)) \land \nu_C(x), \ldots, (\nu_A(x) \lor \nu_B(x)) \land \nu_C(x)) > : x \in X \} = \{ < x : ((\mu_A(x) \lor \mu_B(x)) \land (\mu_A(x) \lor \mu_C(x)), \ldots, (\mu_A(x) \lor \mu_B(x)) \land (\mu_A(x) \lor \mu_C(x)), ((\nu_A(x) \lor \nu_B(x)) \land (\nu_A(x) \lor \nu_C(x), \ldots, (\nu_A(x) \lor \nu_B(x)) \land (\nu_A(x) \lor \nu_C(x)) > : x \in X \} = (A \cup B) \cap (A \cup C)$

Similarly we can prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Also note that $\oplus$ and $\otimes$ are commutative and associative.

The next theorem shows the relations connecting $\cup, \cap, \oplus$, and $\otimes$.

**Theorem 4.9.** For any three IFMSs $A, B, C$

1. $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$
2. $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$

3. $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$

4. $A \otimes (B \cap C) = (A \otimes B) \cap (A \otimes C)$

5. $\neg (A \oplus B) = (\neg A \otimes \neg B)$

6. $\neg (A \otimes B) = (\neg A \oplus \neg B)$

**Proof.** Let $A$, $B$ and $C$ be IFMSs defined as follows.

- $A = \{ x : (\mu^1_A(x), \mu^2_A(x), ..., \mu^p_A(x)), (\nu^1_A(x), \nu^2_A(x), ..., \nu^p_A(x)) > : x \in X \}$
- $B = \{ x : (\mu^1_B(x), \mu^2_B(x), ..., \mu^p_B(x)), (\nu^1_B(x), \nu^2_B(x), ..., \nu^p_B(x)) > : x \in X \}$
- $C = \{ x : (\mu^1_C(x), \mu^2_C(x), ..., \mu^p_C(x)), (\nu^1_C(x), \nu^2_C(x), ..., \nu^p_C(x)) > : x \in X \}$

To prove (1).

Let the $i^{th}$ membership values of an element $x$ in $A$, $B$, $C$ be $\mu^i_A(x), \mu^i_B(x), \mu^i_C(x)$ and the corresponding nonmembership values be $\nu^i_A(x), \nu^i_B(x), \nu^i_C(x)$ respectively.

Then by definition $i^{th}$ membership and nonmembership values of $x$ in $A \oplus (B \cap C)$ be

\[
\text{membership value} = \mu^i_A(x) + (\mu^i_B(x) \wedge \mu^i_C(x)) - \mu^i_A(x)(\mu^i_B(x) \wedge \mu^i_C(x))
\]

\[
\text{nonmembership value} = \nu^i_A(x)(\nu^i_B(x) \lor \nu^i_C(x))
\]

and in $(A \oplus B) \cap (A \oplus C)$ be

\[
\text{membership value} = (\mu^i_A(x) + \mu^i_B(x) - \mu^i_A(x), \mu^i_B(x)) \wedge (\mu^i_A(x) + \mu^i_C(x) - \mu^i_A(x), \mu^i_C(x))
\]

\[
\text{nonmembership value} = \nu^i_A(x)(\nu^i_B(x) \lor \nu^i_C(x))
\]
From the above equations the nonmembership values are coinciding. Now it is enough to check the membership values.

There are two possibilities.

Case (i): $\mu^i_B(x) \geq \mu^i_C(x)$

In this case by putting $a = b = \mu^i_A(x)$, $c = \mu^i_B(x)$ and $d = \mu^i_C(x)$ in Proposition 2 we have

$$i^{th} \text{ membership value of } x \text{ in } A \oplus (B \cap C) \text{ and } (A \oplus B) \cap (A \oplus C)$$

$$= \mu^i_A(x) + \mu^i_C(x) - \mu^i_A(x) \cdot \mu^i_C(x)$$

Case (ii): $\mu^i_B(x) \leq \mu^i_C(x)$

In this case by putting $a = b = \mu^i_A(x)$, $c = \mu^i_C(x)$ and $d = \mu^i_B(x)$ in Proposition 2 we have

$$i^{th} \text{ membership value of } x \text{ in } A \oplus (B \cap C) \text{ and } (A \oplus B) \cap (A \oplus C)$$

$$= \mu^i_A(x) + \mu^i_B(x) - \mu^i_A(x) \cdot \mu^i_B(x)$$

Since the above is true for any element in the membership and nonmembership sequences we can conclude that

$$A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C).$$

(2), (3) and (4) can also be proved by similar arguments.

(5) and (6) follows from the definitions.

$$\square$$

5 Some Operations on Intuitionistic Fuzzy Multisets

First we define Cartesian Product by extending its definition from IFSs.
Definition 5.1. Let
\[ A = \{ < x : (\mu_1^A(x),\mu_2^A(x),\ldots,\mu_p^A(x)),(\nu_1^A(x),\nu_2^A(x),\ldots,\nu_p^A(x)) > : x \in X \} \]
\[ B = \{ < x : (\mu_1^B(x),\mu_2^B(x),\ldots,\mu_p^B(x)),(\nu_1^B(x),\nu_2^B(x),\ldots,\nu_p^B(x)) > : x \in X \} \] be two IFMSs. Then their cartesian product which is denoted by \( A \times B \) is defined as
\[ A \times B = \{ < (x,y) : (\mu_1^A(x)\mu_1^B(y),\mu_2^A(x)\mu_2^B(y),\ldots,\mu_p^A(x)\mu_p^B(y)),(\nu_1^A(x)\nu_1^B(y),\nu_2^A(x)\nu_2^B(y),\ldots,\nu_p^A(x)\nu_p^B(y)) > : x,y \in X \} \].

The following results will ensure that \( A \times B \) is an IFMS.

1. For any real numbers \( a,b,c,d \in [0,1] \) with \( a \geq c \) and \( b \geq d \) it follows that \( a.b \geq c.d \).

2. Also if \( 0 \leq a + b \leq 1 \) and \( 0 \leq c + d \leq 1 \) then
   \[ 0 \leq (a + b)(c + d) \leq 1 \Rightarrow 0 \leq a.c + a.d + b.c + b.d \leq 1 \Rightarrow 0 \leq a.c + b.d \leq 1. \]

Now the following theorem gives the relations connecting Cartesian Product with \( \cup, \cap, \oplus, \otimes \).

Theorem 5.2. For any three IFMSs \( A, B, C \)

1. \( A \times B = B \times A \)
2. \( (A \times B) \times C = A \times (B \times C) \)
3. \( A \times (B \cup C) = (A \times B) \cup (A \times C) \)
4. \( A \times (B \cap C) = (A \times B) \cap (A \times C) \)
5. \( A \times (B \oplus C) \subseteq (A \times B) \oplus (A \times C) \)
6. \( A \times (B \otimes C) \supseteq (A \times B) \otimes (A \times C) \)

Proof. Let \( A, B, \) and \( C \) be IFMSs defined as follows.
\[ A = \{ < x : (\mu_1^A(x),\mu_2^A(x),\ldots,\mu_p^A(x)),(\nu_1^A(x),\nu_2^A(x),\ldots,\nu_p^A(x)) > : x \in X \} \]
\[ B = \{ x : (\mu_B^1(x), \mu_B^2(x), \ldots, \mu_B^p(x)), (\nu_B^1(x), \nu_B^2(x), \ldots, \nu_B^p(x)) > : x \in X \} \]
\[ C = \{ x : (\mu_C^1(x), \mu_C^2(x), \ldots, \mu_C^p(x)), (\nu_C^1(x), \nu_C^2(x), \ldots, \nu_C^p(x)) > : x \in X \} \]

(1) and (2) directly follows from the definition.

To prove (3)
\[ A \times (B \cup C) = \{ (x, y) : (\mu_A^1(x) (\mu_B^1(y) \lor \mu_C^1(y)), \ldots, \mu_A^p(x) (\mu_B^p(y) \lor \mu_C^p(y))), \]
\[ (\nu_A^1(x) (\nu_B^1(y) \land \nu_C^1(y)), \ldots, \nu_A^p(x) (\nu_B^p(y) \land \nu_C^p(y))) > : x, y \in X \} \]
\[ = \{ (x, y) : (\mu_A^1(x) \mu_B^1(y), \ldots, \mu_A^p(x) \mu_B^p(y)), \]
\[ (\nu_A^1(x) \nu_B^1(y), \ldots, \nu_A^p(x) \nu_B^p(y)) > : x, y \in X \} \]
\[ \cup \{ (x, y) : (\mu_A^1(x) \mu_C^1(y), \ldots, \mu_A^p(x) \mu_C^p(y)), \]
\[ (\nu_A^1(x) \nu_C^1(y), \ldots, \nu_A^p(x) \nu_C^p(y)) > : x, y \in X \} \]
\[ = (A \times B) \cup (A \times C) \]

(4) can be proved analogously.

To prove (5)

Let the \( i^{th} \) element in the membership sequence of
\[ A \times (B \oplus C) = \mu_A^i(x) (\mu_B^i(y) + \mu_C^i(y) - \mu_B^i(y) \cdot \mu_C^i(y)) \]
\[ = \mu_A^i(x) \mu_B^i(y) + \mu_A^i(x) \cdot \mu_C^i(y) - \mu_A^i(x) \mu_B^i(y) \cdot \mu_C^i(y) \]
\[ \leq \mu_A^i(x) \mu_B^i(y) + \mu_A^i(x) \cdot \mu_C^i(y) - (\mu_A^i(x))^2 \cdot \mu_B^i(y) \cdot \mu_C^i(y). \]

Now let the \( i^{th} \) element in the nonmembership sequence of
\[ A \times (B \oplus C) = \nu_A^i(x) (\nu_B^i(y) \cdot \nu_C^i(y)) \]
\[ = \nu_A^i(x) \cdot \nu_B^i(y) \cdot \nu_C^i(y) \]
\[ \geq (\nu_A^i(x))^2 \cdot \nu_B^i(y) \cdot \nu_C^i(y). \]

Since the above two inequalities holds for every \( i \) we have
Similarly we can prove (6) also.

\[ A \times (B \oplus C) \subseteq \{ < (x, y) : (\mu_A^1(x) \mu_B^1(y) + \mu_A^1(x) \mu_C^1(y) - \\
\mu_A^1(x)^2, \mu_B^1(y) \mu_C^1(y), ..., \mu_A^p(x) \mu_B^p(y) + \mu_A^p(x) \mu_C^p(y) - \\
\mu_A^p(x)^2, \mu_B^p(y) \mu_C^p(y)), ((\nu_A^1(x))^2, \nu_B^1(y) \nu_C^1(y), \\
..., ((\nu_A^p(x))^2, \nu_B^p(y) \nu_C^p(y)) > : x, y \in X \} \]

\[ = \{ < (x, y) : (\mu_A^1(x) \mu_B^1(y), ..., \mu_A^p(x) \mu_B^p(y)), (\nu_A^1(x) \nu_B^1(y), \\
..., \nu_A^p(x) \nu_B^p(y)) > : x, y \in X \} \oplus \{ < (x, y) : (\mu_A^1(x) \mu_C^1(y), \\
..., \mu_A^p(x) \mu_C^p(y)), (\nu_A^1(x) \nu_C^1(y), ..., \nu_A^p(x) \nu_C^p(y)) > : x, y \in X \} \]

\[ = (A \times B) \oplus (A \times C) \]

\[ \square \]

In [13] a number of cartesian products for IFS are defined, we extend two of them to IFMS.

\[ A \times_2 B = \{ < (x, y) : (\mu_A^1(x) \mu_B^1(y), ..., \mu_A^p(x) \mu_B^p(y)), \\
(\nu_A^1(x) + \nu_B^1(y) - \nu_A^1(x) \nu_B^1(y), ..., \nu_A^p(x) + \nu_B^p(y) - \nu_A^p(x) \nu_B^p(y)) > : x, y \in X \}. \]

\[ A \times_3 B = \{ < (x, y) : min((\mu_A^1(x) \mu_B^1(y)), ..., min(\mu_A^p(x) \mu_B^p(y))), \\
(max(\nu_A^1(x) \nu_B^1(y)), ..., max(\nu_A^p(x) \nu_B^p(y))) > : x, y \in X \}. \]

In [14] another Cartesian Product is introduced for IFS. That also can be extended to IFMS.

\[ A \times_4 B = \{ < (x, y) : ((\mu_A^1(x) + \mu_B^1(y))/2, ..., (\mu_A^p(x) + \mu_B^p(y))/2), \\
((\nu_A^1(x) + \nu_B^1(y))/2, ..., (\nu_A^p(x) + \nu_B^p(y))/2) > : x, y \in X \}. \]

\[ \alpha \beta \text{- cut of an IFMS is introduced in the following definition.} \]
Definition 5.3. Let $A = \{ x : (\mu^1_A(x), \mu^2_A(x), ..., \mu^p_A(x)), (\nu^1_A(x), \nu^2_A(x), ..., \nu^p_A(x)) > : x \in X \}$ be an IFMS. Let $0 \leq \alpha + \beta \leq 1$. Then $\alpha\beta$ -cut of $A$ is the set of all $x$ such that $\mu^j_A(x) \geq \alpha$ with the corresponding $\nu^j_A(x) \leq \beta$ and is denoted by $[A]_{(\alpha,\beta)}$. Clearly it is a crisp multiset.

The count of each element $x \in [A]_{(\alpha,\beta)}$ is obtained using membership and nonmembership sequences from the following algorithm.

**Algorithm 1** Compute the number of occurrence of $x$ in $[A]_{(\alpha,\beta)}$

1: if $\mu^1_A(x) < \alpha$ then
2: count $[A]_{(\alpha,\beta)}(x) = 0$
3: end if
4: if $\mu^j_A(x) \geq \alpha$ and $\mu^{j+1}_A(x) < \alpha$ for some $j < p$ or $\mu^j_A(x) \geq \alpha$ for $j = p$ then
5: arrange the corresponding nonmembership values in ascending order.
6: end if
7: Let it be $(\nu^1_A)^1(x) \leq (\nu^1_A)^2(x) \leq ... \leq (\nu^1_A)^j(x); j \leq p$
8: if $(\nu^1_A)^1(x) > \beta$ then
9: count $[A]_{(\alpha,\beta)}(x) = 0$
10: end if
11: if $(\nu^1_A)^i(x) \leq \beta$ and $(\nu^1_A)^{i+1}(x) > \beta$ for some $i < j$ or $(\nu^1_A)^i(x) \leq \beta$ for $i = j$ then
12: count $[A]_{(\alpha,\beta)}(x) = i$
13: end if

Definition 5.4. Let $A = \{ x : (\mu^1_A(x), \mu^2_A(x), ..., \mu^p_A(x)), (\nu^1_A(x), \nu^2_A(x), ..., \nu^p_A(x)) > : x \in X \}$ be an IFMS. Let $0 \leq \alpha + \beta \leq 1$. Then strong $\alpha\beta$ -cut of $A$ is the set of all $x$ such that $\mu^j_A(x) > \alpha$ with the corresponding $\nu^j_A(x) < \beta$ and is denoted by $[A]_{(\alpha,\beta)}^*$. It is also a crisp multiset.
The count of each element \( x \in [A]_{(\alpha,\beta)} \) can be obtained in the same way.

**Example 5.5.** Let

\[
A = \{ < x : (0.4, 0.2), (0.5, 0.3) >, < y : (0.5, 0.5, 0.3), (0.5, 0.5, 0.2) >, \\
< z : (0.6, 0.6, 0.5, 0.3), (0.4, 0.3, 0.5, 0.2) >, < w : (0.7, 0.6, 0.5), (0.3, 0.2, 0.4) > \}
\]

Also let \( \alpha = 0.5, \beta = 0.4 \).

Then \([A]_{(\alpha,\beta)} = \{ 2 | z, 3 | w \}, [A]_{(\alpha,\beta)}^* = \{ 1 | z, 2 | w \}\)

The next theorem is an immediate consequence of the above definitions.

**Theorem 5.6.** Let \( A \) and \( B \) are two IFMSs drawn from a set \( X \). Then \( A \subseteq B \)
if and only if \([A]_{(\alpha,\beta)} \subseteq [B]_{(\alpha,\beta)} \) for every \( \alpha, \beta \in [0,1] \) with \( 0 \leq \alpha + \beta \leq 1 \).

**Proof.** Suppose \( A \subseteq B \).

Let \( x \in [A]_{(\alpha,\beta)} \) with count\([A]_{(\alpha,\beta)}(x) = m \) where \( m \in \{1, 2, \ldots, L(x)\} \).

Then by definition

\[
\mu^m_A(x) \geq \alpha, \ (\nu'_A)^m(x) \leq \beta \tag{5.1}
\]

Since \( A \subseteq B \), by definition

\[
\mu_j^j_A(x) \leq \mu_j^j_B(x) \text{ and } \nu_j^j_A(x) \geq \nu_j^j_B(x) \text{ for every } j \leq L(x).
\]

Hence if we go through the algorithm described in definition 5.3 for finding \( \alpha\beta \)
-cut of \( B \) (after arranging the nonmembership values in ascending order) we can say that

\[
\mu_j^j_A(x) \leq \mu_j^j_B(x), \ (\nu'_A)^j(x) \geq (\nu'_B)^j(x) \tag{5.2}
\]

for every \( j \).

From equations 5.1 and 5.2

\[
\mu^m_B(x) \geq \alpha \text{ and } (\nu'_B)^m(x) \leq \beta \text{ for } m \in \{1, 2, \ldots, L(x)\}
\]

\( \Rightarrow x \in [B]_{(\alpha,\beta)} \) with count\([B]_{(\alpha,\beta)}(x) \geq m \).
\[ A(\alpha, \beta) \subseteq [B](\alpha, \beta) \]

Conversely suppose that \[ A(\alpha, \beta) \subseteq [B](\alpha, \beta) \] for every \( \alpha, \beta \in [0, 1] \) with \( 0 \leq \alpha + \beta \leq 1 \).

So, in particular, for \( \alpha = 0 \) and \( \beta = 1 \) we have \[ A(\alpha, \beta) = A \] and \[ [B](\alpha, \beta) = B \].

Hence the proof. \( \square \)

Next we define two operators similar to ‘Necessity’ and ‘Possibility’ in Modal logic. The two operators transform every IFMS into FMS.

If \( A = \{ x : (\mu_1^A(x), \mu_2^A(x), ..., \mu_p^A(x)), (\nu_1^A(x), \nu_2^A(x), ..., \nu_p^A(x)) > : x \in X \} \) be an IFMS then

\[ \Box A = \{ x : (\mu_1^A(x), ..., \mu_p^A(x)), (1 - \mu_1^A(x), ..., 1 - \mu_p^A(x)) > : x \in X \} \]

\[ = \{ x : (\mu_1^A(x), ..., \mu_p^A(x)) > : x \in X \} \]

\[ \Diamond A = \{ x : (1 - \nu_1^A(x), ..., 1 - \nu_p^A(x), (\nu_1^A(x), ..., \nu_p^A(x))) > : x \in X \} \]

\[ = \{ x : (1 - \nu_1^A(x), ..., 1 - \nu_p^A(x)) > : x \in X \} \]

The n-ary possibility and n-ary necessity is defined as

\[ \Box^n(A) = \Box^{n-1}(\Box(A)) = \Box((...\Box(\Box(A))))(\Box \text{ operates } n \text{ times}) \]

\[ \Diamond^n(A) = \Diamond^{n-1}(\Diamond(A)) = \Diamond((...\Diamond(\Diamond(A))))(\Diamond \text{ operates } n \text{ times}) \]

In the ordinary IFS case, the two operators are introduced in [2] and the statement analogous to the following Lemma is proved.

**Theorem 5.7.** For every IFMS A

1. \( \Box^n(A) = \Box A \) for every positive integer \( n \).

2. \( \Diamond^n(A) = \Diamond A \) for every positive integer \( n \).
3. $\square \Diamond A \neq \Diamond \square A$.

The proof follows from the definition.

**Theorem 5.8.** For any two IFMSs $A$ and $B$

1. $\square (\neg A) = \neg (\Diamond A)$
2. $\Diamond (\neg A) = \neg (\square A)$
3. $\square (A \cup B) = \square A \cup \square B$
4. $\Diamond (A \cup B) = \Diamond A \cup \Diamond B$
5. $\square (A \cap B) = \square A \cap \square B$
6. $\Diamond (A \cap B) = \Diamond A \cap \Diamond B$

**Proof.** Let $A = \{ x : (\mu_A(x), \mu_2(x), ..., \mu_p(x)) > x \in X \}$ and $B = \{ x : (\mu_B(x), \mu_2(x), ..., \mu_p(x)) > x \in X \}$.

(1) and (2) follows from the definition.

To prove (3)

\[
\square (A \cup B) = \square \{ x : (\mu_A(x) \vee \mu_B(x), ..., \mu_A(x) \vee \mu_B(x)) > x \in X \}
\]
\[
= \{ x : (\mu_A(x) \vee \mu_B(x), ..., \mu_A(x) \vee \mu_B(x)) \wedge (1 - \mu_A(x) \vee \mu_B(x)) > x \in X \}
\]
\[
= \{ x : (\mu_A(x) \vee \mu_B(x), ..., \mu_A(x) \vee \mu_B(x)), ((1 - \mu_A(x) \wedge (1 - \mu_B(x))) > x \in X \}
\]
\[
= \{ x : (\mu_A(x), ..., \mu_A(x)), (1 - \mu_A(x), ..., 1 - \mu_A(x)) > x \in X \} \cup \{ x : (\mu_B(x), ..., \mu_B(x)), (1 - \mu_B(x), ..., 1 - \mu_B(x)) > x \in X \}
\]
\[
= \square A \cup \square B.
\]
To prove (6)

\[ \Diamond(A \cap B) = \{ < x : (\mu_1^A(x) \land \mu_1^B(x), ..., \mu_1^p(x) \land \mu_1^p(x)), (\nu_1^A(x) \lor \nu_1^B(x), ..., \nu_1^p(x) \lor \nu_1^p(x)) > : x \in X \} \]

\[ = \{ < x : (1 - (\nu_1^A(x) \lor \nu_1^B(x)), ..., 1 - (\nu_1^p(x) \lor \nu_1^p(x))), ((\nu_1^A(x) \lor \nu_1^B(x)), ..., (\nu_1^p(x) \lor \nu_1^p(x))) > : x \in X \}, \]

\[ = \{ < x : ((1 - \nu_1^1(x)) \land (1 - \nu_1^B(x)), ..., (1 - \nu_1^p(x)) \land (1 - \nu_1^p(x)), (\nu_1^1(x)) \lor (\nu_1^B(x)), ..., (\nu_1^p(x)) \lor (\nu_1^p(x))) > : x \in X \} \]

\[ \cap \{ < x : (1 - \nu_1^1(x), ..., 1 - \nu_1^p(x)), (\nu_1^1(x), ..., \nu_1^p(x)) > : x \in X \} \]

\[ = \Diamond A \cap \Diamond B. \]

(4) and (5) can be proved analogously.

\[ \square \]

6 IFMS and Intuitionistic Fuzzy Multi-dimensional set

**Definition 6.1.** [15] Let set \( E \) be fixed. Let sets \( Z_1, Z_2, ..., Z_n \) be fixed and let for each \( i(1 \leq i \leq n) : z_i \in Z_i \). An IFS \( A \) is an object of the *Intuitionistic Fuzzy Multi-dimensional set* (IFMDS) \( A \) in \( E, Z_1, Z_2, ..., Z_n \) is an object of the form \( A(Z_1, Z_2, ..., Z_n) = \{ < x : \mu_A(x, z_1, z_2, ..., z_n), \nu_A(x, z_1, z_2, ..., z_n) > : \}

\[ < x, z_1, z_2, ..., z_n > \in E \times Z_1 \times Z_2 \times ... \times Z_n \}, \text{ where} \]

\[ (a) \mu_A(x, z_1, z_2, ..., z_n) + \nu_A(x, z_1, z_2, ..., z_n) \leq 1 \text{ for every } < x, z_1, z_2, ..., z_n > \in E \times Z_1 \times Z_2 \times ... \times Z_n, \]
(b) \( \mu_A(x, z_1, z_2, ..., z_n) \) and \( \nu_A(x, z_1, z_2, ..., z_n) \) are the degrees of membership and non-membership, respectively, of the element \( x \in E \) and \( < z_1, z_2, ..., z_n > \in Z_1 \times Z_2 \times ... \times Z_n \).

**Remark 6.2.** The authors in the previous definition defined IFMDS as a new extension of Intuitionistic Fuzzy sets. Here \((z_1, z_2, ..., z_n)\) is considered as a n-dimensional vector. Thus for each \( x \in E \), \( A(Z_1, Z_2, ..., Z_n) \) forms a multiset with membership and nonmembership values. In IFMS concept, we combined the two concept IFS and FMS. Since in FMS, membership values are arranged in decreasing order, we adopt the same criteria by defining

\[
\mu_1^A(x) \geq \mu_2^A(x) \geq \ldots \geq \mu_p^A(x)
\]

for each \( x \in X \). This is the main difference between the two concepts. Detailed discussion of IFMDS can be seen in [16] and [17].

### 7 Conclusion

In this paper, the various basic operations, definitions and theorems related to Intuitionistic Fuzzy Multiset have been discussed. There is an excellent opportunity for further research in theoretical as well as application point of view. Many existing results from IFS and FMS can be extended to IFMS. The concept of Intuitionism can be applied in Information retrieval, Flexible querying etc. Also the different forms of the De Morgan Laws can be seen in [18]. A short comparison of IFMS and IFMDS is also discussed. There is scope for a detailed analysis between these two concepts.

### References


