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A Three-Degree of Freedom Mathematical Model Simulating Free Vibration of a Prismatic Reinforced Concrete Pile in Axial Motion

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Abstract:* In this paper, a three degree of freedom mathematical model is formulated to simulate the free vibration of a prismatic reinforced concrete pile in axial motion using flexibility method of structural analysis. The reinforced concrete pile itself is analyzed as a flexural beam with distributed mass and elasticity. The distributed mass is assumed to be lumped at three discrete points called the nodal points and the displacements at the nodal points of the lumped masses are obtained using slope and deflection method of structural analysis with the unit load serving as the transverse load. The lumped masses are assumed to undergo oscillation about their mean position in the direction transverse to the longitudinal axis of the weightless beam. The flexural flexibility and axial flexibility factor are assumed to be dynamically equivalent. Piles undergo axial motion under repeated loading from impact hammer. The natural frequencies of vibration for the three vibration modes were compared with the simulated values obtained from literature to demonstrate the effectiveness of the formulated models and were found to agree favorably showing that the formulated model can be used to simulate the free vibration of a prismatic reinforced concrete pile in axial motion.

Keywords: **Three-degree of freedom, mathematical model, free vibration, axial motion, simulate.**

I. INTRODUCTION

Piles are used as foundation elements and are employed where and when heavy structural loads are required to be transmitted to the load bearing stratum of the soil. Vibration of piles due to repeated blows from impact hammer is axial in nature and therefore, produces axial deformation which results from axial motion of the pile. The dynamic analysis of free vibration of a prismatic reinforced concrete pile in both cohesive and cohesion less soils is a task of paramount importance to structural engineers as it may lead to a very dangerous phenomenon called resonance when the natural frequency of any of the vibration modes is exceeded by the excitation frequency resulting from repeated blows on the pile [1] – [4]. The prismatic reinforced concrete pile is a flexural beam with distributed mass and elasticity and all structural systems having distributed mass and elasticity posses infinite number of degrees of freedom [5] – [6]. Degree of freedom is the number of independent geometric parameters that describe the positions of all masses for all possible displacements of the structural system at any moment in time. For instance, for a cantilevered beam with evenly distributed mass, an infinite number of coordinates are needed to define the displacement configuration. If the beam mass is lumped into two bodies the force-displacement properties of the cantilever remaining constant and the external forces causing the motion are applied at the two masses, then the deflected configuration of the beam at any moment in time can be completely determined by 12 coordinates with 6 coordinates at each lumped masses [7].The dynamic analysis of structural systems with infinite number of degrees of freedom is difficult and mathematically cumbersome [8] – [9]. In this paper, a mathematical model is formulated to simulate the free vibration of a prismatic reinforced concrete pile in axial motion using flexibility and slope and deflection methods of structural analysis. The formulated model is simple and straightforward.

II. MATHEMATICAL FORMULATION

Consider a three-degrees of freedom weightless flexural beam subjected to a system of unit loads P_1, P_2 and P_3 at coordinates 1, 2 and 3 as shown in Fig. 1. The displacements $X_1(t),\ X_2(t)\ and\ X_3(t)$ at the respective coordinates due to masses $m_1, m_2\ and\ m_3$ are as follows:

$$X_1(t) = -f_{11}m_1\ddot{X}_1 + \left(-f_{12}m_2\ddot{X}_2\right) + \left(-f_{13}m_3\ddot{X}_3\right) \quad (1)$$

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$$X_2(t) = -f_{21}m_1\ddot{X}_1 + \left(-f_{22}m_2\ddot{X}_2\right) + \left(-f_{23}m_3\ddot{X}_3\right) \quad (2)$$

$$X_3(t) = -f_{31}m_1\ddot{X}_1 + \left(-f_{32}m_2\ddot{X}_2\right) + \left(-f_{33}m_3\ddot{X}_3\right) \quad (3)$$

Where:

$f_{ij} =$ flexibility factor resulting from unit load

$m_i =$ An ith lumped mass at ith nodal point.

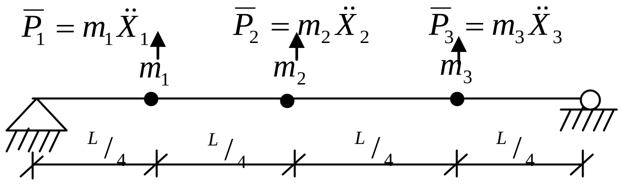

Fig 1: A 3 - degree of freedom weightless beam model.

The generalized equation of motion considering the three lumped masses m_1, m_2 *and* m_3 is:

$$X_j(t) + \sum_{i=1}^{3} f_{ij}\, m_j\, X_j = 0 \quad (4)$$

The solution to equation (4) can be in the form:

$$X_j(t) = C_j \cos wt \quad (5)$$

Where:

$C_j\,(j = 1, 2, 3)$ represent the amplitudes of displacement due to m_1, m_2 *and* m_3 respectively.

From equation (4), the force of inertia due to mass m_1, m_2 *and* m_3 in generalized form is given by:

$$F_j = -m_j\,\ddot{X}_j \quad (6)$$

$(j = 1, 2, 3)$

The second derivative of equation (5) with respect to as is:

$$X_j^{''}(t) = -C_j w^2 \cos wt \quad (7)$$

Substituting for X_j in equation (6) transforms equation (6) to:

$$F_j = -m_j C_j w^2 \cos wt \quad (8)$$

According to Osadebe [8],

The amplitude of the mass m_j at ith nodal point is given by:

$$y_j = m_j C_j w^2 \quad (9)$$

Therefore,

$$C_j = \frac{y_j}{m_j w_j^2} \quad (10)$$

Where:

$w_j =$ natural frequency of vibration associated with a particular vibration mode.

Substituting for $X_j(t)$ in equation (4) and using equation (10) transforms equation (4) to:

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$$\left[\frac{y_j}{m_j w_j^2} - \sum_{i=1}^{3} f_{ij}\, y_j\right]\cos wt = 0 \quad (11)$$

$(j = 1, 2, 3)$

$$\cos wt = 0 \quad (12)$$

The generalized frequency equation now becomes:

$$\frac{y_j}{m_j w_j^2} - \sum_{i=1}^{3} f_{ij}\, y_j = 0 \quad (13)$$

At points $j = 1, 2\, and\ 3$ using equation (13), the following equations are obtained:

$$f_{11}y_1 + f_{12}y_2 + f_{13}y_3 = 0 \quad (14)$$

$$f_{21}y_1 + f_{22}y_2 + f_{23}y_3 = 0 \quad (15)$$

$$f_{31}y_1 + f_{32}y_2 + f_{33}y_3 = 0 \quad (16)$$

Where:

$$\left.\begin{aligned} f_{11} &= f_{11} - \left(m_1 w_1^2\right)^{-1} \\ f_{22} &= f_{22} - \left(m_2 w_2^2\right)^{-1} \\ f_{33} &= f_{33} - \left(m_3 w_3^2\right)^{-1} \end{aligned}\right\} \quad (17)$$

Putting equations (14 – 16) in matrix form we have:

$$\begin{bmatrix} f_{11} - \left(m_1 w_1^2\right)^{-1} & f_{12} & f_{13} \\ f_{21} & f_{22} - \left(m_2 w_2^2\right)^{-1} & f_{23} \\ f_{31} & f_{32} & f_{33} - \left(m_3 w_3^2\right)^{-1} \end{bmatrix} = 0 \quad (18)$$

Let

$$\lambda = \left(m_j w_j^2\right)^{-1} \quad (19)$$

Then, from equation (18) we have:

$$\begin{bmatrix} f_{11} - \lambda_1 & f_{12} & f_{13} \\ f_{21} & f_{22} - \lambda_2 & f_{23} \\ f_{31} & f^{32} & f_{33} - \lambda_3 \end{bmatrix} = 0 \quad (20)$$

Expansion of equation (20) gives $\lambda_1, \lambda_2, ..., \lambda_n$ such that:

$$\lambda_1 < \lambda_2 < \lambda_3 \quad (21)$$

III. STRUCTURAL MODEL

The reinforced concrete pile behaves structurally as a flexural beam. Consider a flexural beam subjected to 3-degrees of freedom lumped masses as shown in Fig.1 and carrying a point load p at the nodal points 1, 2 and 3 respectively.

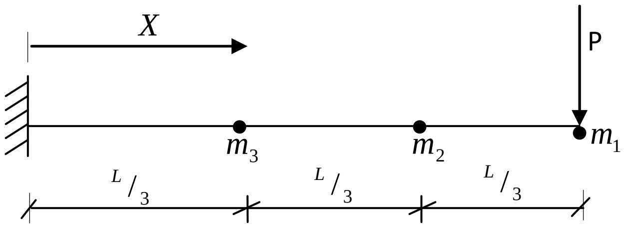

Fig 2: Derivation of flexibility factors for the three locations of the lumped masses

From structural analysis, the generalized displacement equation for flexural beam subjected to a concentrated end load is given by:

$$y = \frac{P(L-X)^3}{6EI} + \frac{PL^2}{2EI}X - \frac{PL^3}{6EI} \tag{22}$$

Where:

$X =$ location of a particular inertia mass at a particular nodal point

Using unit load method implies that P =1 at each nodal point.

Equation (22) now transforms to:

$$f_{ij} = \frac{(L-X)^3}{6EI} + \frac{L^2 X}{2EI} - \frac{L^3}{6EI} \tag{23}$$

IV. RESULTS AND DISCUSSION

An example problem

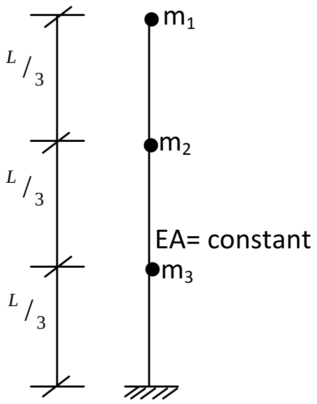

Fig 3 (a): A Cantilever model

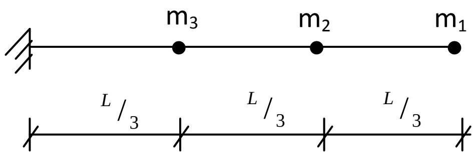

Fig 3(b): A flexural beam model

The flexibility factors obtained using equation (23) are as follows:

$$f_{11} = \frac{L^3}{3E1},\ f_{22} = \frac{8L^3}{81EI},\ f_{33} = \frac{L^3}{81EI}$$

According to Maxwell's Reciprocal Theorem,

$$f_{12} = f_{21} = \frac{14L^3}{81EI},\ f_{23} = f_{23} = \frac{2.5L^3}{81EI},\ f_{13} = f_{31} = \frac{4L^3}{81EI}$$

The flexural flexibility factor $\frac{L^3}{EI}$ is assumed to be dynamically equivalent to axial flexibility factor $\frac{L}{AE}$. Putting the obtained axial flexibility factors in matrix form using equation (20) we have:

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$$\begin{bmatrix} 0.333\frac{L}{AE}-\lambda & \frac{0.173L}{AE} & \frac{0.049L}{AE} \\ 0.173\frac{L}{AE} & 0.099\frac{L}{AE}-\lambda & 0.031\frac{L}{AE} \\ 0.049\frac{L}{AE} & 0.031\frac{L}{AE} & 0.012\frac{L}{AE}-\lambda \end{bmatrix} = 0 \quad (24)$$

Multiplying the first, second and third row by axial rigidity AE and dividing the same row 1, 2 and 3 by L we have:

$$\begin{bmatrix} 0.333\frac{\lambda AE}{L} & 0.173 & 0.049 \\ 0.173 & 0.099-\frac{\lambda AE}{L} & 0.031 \\ 0.049 & 0.031 & 0.012-\frac{\lambda AE}{L} \end{bmatrix} = 0 \quad (25)$$

Again, let

$$\frac{\lambda AE}{L}=\beta \quad (26)$$

Equation (25) now transforms to:

$$\begin{bmatrix} 0.333-\beta & 0.173 & 0.049 \\ 0.173 & 0.099-\beta & 0.031 \\ 0.049 & 0.031 & 0.012-\beta \end{bmatrix} = 0 \quad (27)$$

Expansion of the determinant in equation (27) and simplifying yields the polynomial:

$$\beta^3 - 0.444\beta^2 + 0.00483\beta - 0.000005 = 0 \quad (28)$$

The roots are:

$$\beta_1 = 0.266,\ \beta_2 = 0.022,\ \beta_3 = 0.058$$

Considering figure 3(b) and from statics,

$$m_1 = \frac{2\rho l}{3},\ m_2 = \frac{4\rho l}{3},\ m_3 = \frac{4\rho l}{3}$$

Where:

ρ = mass per unit length.

Using equation (26),

$$\frac{\lambda AE}{L} = 0.266$$

$$\Rightarrow \quad \frac{AE}{\frac{2}{3}\rho L\omega_1^2 L} = 0.266$$

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$$\Rightarrow \quad \omega_1 = \frac{1.583}{l}\sqrt{\frac{AE}{\rho}}\ rad/\sec$$

$$\frac{\lambda AE}{L} = 0.058$$

$$\therefore \frac{AE}{\frac{4}{3}\rho L^2 \omega_2} = 0.022$$

$$\omega_2 = \frac{4.795}{L}\sqrt{\frac{AE}{L}}\ rad/\sec$$

$$\frac{\lambda AE}{L} = \beta_3 = 0.058$$

$$\Rightarrow \quad \frac{AE}{\frac{4}{3}\rho L^2 \omega_3^2} = 0.058$$

$$\Rightarrow \omega_3 = \frac{7.785}{L}\sqrt{\frac{AE}{L}}\ rad/s$$

Table I: Comparison of the results of present model with those of the exact solution

Natural frequency (rad/s)	Present model	Exact solution [10]
w_1	$\frac{1.583}{L}\sqrt{\frac{AE}{\rho}}$	$\frac{1.571}{L}\sqrt{\frac{AE}{\rho}}$
w_2	$\frac{4.795}{L}\sqrt{\frac{AE}{\rho}}$	$\frac{4.713}{L}\sqrt{\frac{AE}{\rho}}$
w_3	$\frac{7.785}{L}\sqrt{\frac{AE}{\rho}}$	$\frac{7.855}{L}\sqrt{\frac{AE}{\rho}}$

V. Discussion of Results and Conclusion

The results of simulation of 3-degrees of freedom reinforced concrete pile in axial motion is as presented in Table I. From Table I, it can be seen that the present formulation yields results that agreed favorably with those of the exact solution. The disparity between the results of the present model and those of the exact values may be due to model errors. It can be seen that the present model yielded results that are almost identical with those of the simulated values showing the effectiveness of the present model in the analysis of dynamic response of a prismatic reinforced concrete pile in axial motion.

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