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Crank-Nicolson Fully Discrete Positive Definite Expanded Mixed Method for Parabolic Integro-Differential Equations

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***Abstract*—A Crank-Nicolson fully discrete positive definite expanded mixed finite element method is studied for second-order parabolic partial integro-differential equations. Compared to Chen's expanded mixed method, the studied system is symmetric positive definite and both the gradient equation and the flux equation are separated from its scalar unknown equation. Some a priori error estimates for three variables based on Crank-Nicolson fully discrete scheme are obtained.**

***Index Terms*—parabolic integro-differential equations, Positive definite expanded mixed element method, Crank-Nicolson scheme, fully discrete error estimates.**

I. INTRODUCTION

In this article, we consider the following parabolic partial integro-differential equation [1-8]

$$\begin{cases} u_t - \nabla\cdot(a(x,t)\nabla u + b(x,t)\int_0^t \nabla u ds) = f(x,t), & (x,t)\in\Omega\times J, \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times\bar{J}, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded convex polygonal domain in $R^d, (d=1,2,3)$, $J=(0,T]$ is the time interval with $0<T<\infty$. The coefficients $a=a(x,t), b=b(x,t)$ are two functions, which satisfy the property that there exists some positive constants $a_{\min}, a_{\max}, b_{\min}$, and $b_{\max}$ such that $0<a_{\min}\le a(x,t)\le a_{\max}$ and $0<b_{\min}\le b(x,t)\le b_{\max}$.

In 1994, Chen [9] proposed an expanded mixed element method for second-order linear elliptic equation. Compared to standard mixed element methods, the expanded mixed method is expanded in the sense that three variables are explicitly approximated, namely, the scalar unknown, its gradient, and its flux. Since then, many researchers have solved a lot of partial differential equations [10-15] based on Chen's expanded mixed method [9].

In 2001, Yang [16] proposed and analyzed a splitting positive definite mixed procedure, which does not lead to some saddle point problems, to treat the pressure equation of parabolic type in a nonlinear parabolic system. From then on, the method has been solved many problems, such as hyperbolic equation [17], pseudo-hyperbolic equations [18], viscoelastic wave equation [19,20], integro-differential equations [21,22].

In 2012, Liu et al. [23] proposed a new backward Euler positive definite expanded mixed method based on expanded mixed method [9] and splitting mixed element systems [16]. In this article, our purpose is to analyze a Crank-Nicolson fully discrete scheme for parabolic integro-differential equations based on the positive expanded mixed systems [23]. The proposed mixed scheme is symmetric positive definite and avoids some saddle point problems. What's more, both the gradient equation and the flux equation are separated from its scalar unknown equation. Some a priori error estimates are derived for both Crank-Nicolson fully discrete schemes.

Throughout this article, C will denote a generic positive constant independent of the space-time mesh parameters h_u, h_σ, and δ. Usual definitions, notations, and norms of Sobolev spaces as in Ref. [24,25] are used. The natural inner product in $L^2(\Omega)$ or $[L^2(\Omega)]^d$ is defined by $(\cdot,\cdot)$ with norm $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{\mathbf{L}^2(\Omega)}$.

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II. A NEW EXPANDED MIXED VARIATIONAL FORMULATION

Introducing the auxiliary variables

$$\lambda=\nabla u\ ,\sigma=a(x,t)\nabla u+b(x,t)\int_0^t\nabla u ds=a\lambda+b\int_0^t\lambda ds. \tag{2.1}$$

Then, we obtain the equivalent system of parabolic partial integro-differential equations for the problem (1.1)

$$\begin{cases}(a)u_t-\nabla\cdot\sigma=f(x,t), & (x,t)\in\Omega\times J,\\ (b)\lambda-\nabla u=0, & (x,t)\in\Omega\times J,\\ (c)\sigma-a\lambda-b\int_0^t\lambda ds=0, & (x,t)\in\Omega\times J,\end{cases} \tag{2.2}$$

with the initial values $\lambda(x,0)=\nabla u_0(x)$, $\sigma(x,0)=a\nabla u_0(x)$, and $u(x,0)=u_0(x)$.

Then, an expanded mixed weak formulation of (2.2) can be obtained by

$$\begin{cases}(a)(u_t,v)-(\nabla\cdot\sigma,v)=(f(x,t),v), & \forall v\in L^2(\Omega),\\ (b)(\lambda,\mathbf{w})+(u,\nabla\cdot\mathbf{w})=0, & \forall w\in\mathbf{W},\\ (c)(\sigma,\mathbf{z})-(a\lambda,\mathbf{z})-(b\int_0^t\lambda ds,\mathbf{z})=0, & \forall z\in\mathbf{W}.\end{cases} \tag{2.3}$$

Where $\mathbf{W}=H(\text{div};\Omega)=\{\omega\in[L^2(\Omega)]^d;\nabla\cdot\omega\in L^2(\Omega)\}$.

From (2.3b) we derive

$$(\lambda_t,\mathbf{w})+(u_t,\nabla\cdot\mathbf{w})=0. \tag{2.4}$$

Taking $v=\nabla\cdot\mathbf{w}$ in (2.3a) for $\mathbf{w}\in\mathbf{W}$ and then substituting it into (2.4),we derive a new equivalent weak formulation of the system (2.3):

$$\begin{cases}(a)(\lambda_t,\mathbf{w})+(\nabla\cdot\sigma,\nabla\cdot\mathbf{w})=-(f(x,t),\nabla\cdot\mathbf{w}), & \forall\mathbf{w}\in\mathbf{W},\\ (b)(\sigma,\mathbf{z})-(a\lambda,\mathbf{z})-(b\int_0^t\lambda ds,\mathbf{z})=0, & \forall\mathbf{z}\in\mathbf{W},\\ (c)(u_t,v)-(\nabla\cdot\sigma,v)=(f(x,t),v), & \forall v\in L^2(\Omega).\end{cases} \tag{2.5}$$

Let Γ_{h_u} and Γ_{h_σ} be two families of quasi-regular partions of the domain Ω, which may be the same one or not, such that the elements in the partitions have the diameters bounded by h_u and h_σ,respectively. Let $X_{h_u}\subset L^2(\Omega)$ and $\mathbf{V}_{h_\sigma}\subset\mathbf{W}$ be finite element spaces defined on the partitions Γ_{h_u} and Γ_{h_σ}.

Now the semidiscrete positive definite expanded mixed finite element method for (2.5) consists in determining $(u_h,\lambda_h,\sigma_h)\in X_{h_u}\times\mathbf{V}_{h_\sigma}\times\mathbf{V}_{h_\sigma}$ such that

$$\begin{cases}(a)(\lambda_{ht},\mathbf{w}_h)+(\nabla\cdot\sigma_h,\nabla\cdot\mathbf{w}_h)=-(f(x,t),\nabla\cdot\mathbf{w}_h), & \forall\mathbf{w}_h\in\mathbf{V}_{h_\sigma},\\ (b)(\sigma_h,\mathbf{z}_h)-(a\lambda_h,\mathbf{z}_h)-(b\int_0^t\lambda_h ds,\mathbf{z}_h)=0, & \forall\mathbf{z}_h\in\mathbf{V}_{h_\sigma},\\ (c)(u_{ht},v_h)-(\nabla\cdot\sigma_h,v_h)=(f(x,t),v_h), & \forall v_h\in X_{h_u},\end{cases} \tag{2.6}$$

III. FULLY DISCRETE ERROR ESTIMATES

A. Some projections

Let X_{h_u} and $\mathbf{V}_{h_\sigma}$ be finite dimensional subspaces of $L^2(\Omega)$ and $\mathbf{W}$ respectively, with the inverse property (see Ref. [25]) and the following approximation properties (see Ref. [26-28]): for $0\le p\le+\infty$ and r,r^*,k positive integers

$$\begin{aligned}&\inf_{\mathbf{w}_h\in\mathbf{V}_{h_\sigma}}\|\mathbf{w}-\mathbf{w}_h\|_{\mathbf{L}^p(\Omega)}\le Ch_\sigma^{r+1}\|\mathbf{w}\|_{\mathbf{W}^{r+1,p}(\Omega)},\forall\mathbf{w}\in H(\text{div};\Omega)\cap[W^{r+1,p}(\Omega)]^d,\\ &\inf_{\mathbf{w}_h\in\mathbf{V}_{h_\sigma}}\|\nabla\cdot(\mathbf{w}-\mathbf{w}_h)\|_{L^p(\Omega)}\le Ch_\sigma^{r^*}\|\nabla\cdot\mathbf{w}\|_{W^{r^*,p}(\Omega)},\forall\mathbf{w}\in H(\text{div};\Omega)\cap[W^{r+1,p}(\Omega)]^d,\\ &\inf_{v_h\in X_{h_u}}\|v-v_h\|_{L^p(\Omega)}\le Ch_u^{k+1}\|v\|_{W^{k+1,p}(\Omega)},\forall v\in L^2(\Omega)\cap W^{k+1,p}(\Omega),\end{aligned} \tag{3.1}$$

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Where $r^* = r+1$ for the Brezzi-Douglas-Fortin-Marini spaces[28], $r^* = r$ for the Brezzi-Douglas-Marini spaces[26].

For a priori error analysis, we introduce two operators. It is well known that, there exists an operator R_h from $H(\text{div};\Omega)$ onto $\mathbf{V}_{h_\sigma}$,see Ref. [26-28], such that, for $1 \le p \le +\infty$,

$$\begin{aligned}
&(\nabla\cdot(\sigma - R_h\sigma), \phi_h) = 0, \forall \phi_h \in \nabla\cdot\mathbf{V}_{h_\sigma} = \{\phi_h = \nabla\cdot\mathbf{w}_h, \mathbf{w}_h \in \mathbf{V}_{h_\sigma}\};\\
&\|\sigma - R_h\sigma\|_{\mathbf{L}^p(\Omega)} \le Ch_\sigma^{r+1}\|\sigma\|_{\mathbf{W}^{r+1,p}(\Omega)};\\
&\|\nabla\cdot(\sigma - R_h\sigma)\|_{L^p(\Omega)} \le Ch_\sigma^{r^*}\|\nabla\cdot\sigma\|_{W^{r^*,p}(\Omega)}.
\end{aligned} \tag{3.2}$$

We also introduce a L^2-project operator P_h from $L^2(\Omega)$ onto X_{h_u} such that

$$\begin{aligned}
&(v - P_h v, v_h) = 0, \forall v \in L^2(\Omega), v_h \in X_{h_u};\\
&\|v - P_h v\|_{L^p(\Omega)} \le Ch_u^{k+1}\|v\|_{H^{k+1,p}(\Omega)}, \forall v \in H^{k+1}(\Omega).
\end{aligned} \tag{3.3}$$

B. Error estimates

For the Crank-Nicolson procedure, let $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ be a given partition of the time interval $[0,T]$ with step length $\delta = T/M$, for some positive integer M . For a smooth function ϕ on $[0,T]$, define $\phi^n = \phi(t_n)$ and $\partial_t\phi^n = (\phi^n - \phi^{n-1})/\delta$, $\phi^{n-\frac{1}{2}} = \dfrac{\phi(t_n)+\phi(t_{n-1})}{2}$ and $\partial_t\phi^{n-\frac{1}{2}} = (\phi^n - \phi^{n-2})/2\delta$

For approximating the integrals, we use the compound trapezoidal rule

$$\frac{\delta}{2}\sum_{j=0}^{n-1}(\phi^j + \phi^{j+1}) \approx \int_0^{t_n}\phi(s)ds.$$

Note that $\phi \in C^1[0,T]$, the quadrature error satisfies

$$\left|\frac{\delta}{2}\sum_{j=0}^{n-1}(\phi^j + \phi^{j+1}) - \int_0^{t_n}\phi(s)ds\right| \le C\delta^2\int_0^{t_n}|\phi_s(s)|ds$$

The Eq.(2.5) has the following equivalent formulation

$$\begin{cases}
(a)(\partial_t\lambda^n, \mathbf{w}) + (\nabla\cdot\sigma^{n-\frac{1}{2}}, \nabla\cdot\mathbf{w}) = (-f^{n-\frac{1}{2}}, \nabla\cdot\mathbf{w}) + (R_1^n, \mathbf{w}), & \forall\mathbf{w}\in\mathbf{W},\\
(b)(\sigma^{n-\frac{1}{2}}, \mathbf{z}) - (\dfrac{a^n\lambda^n + a^{n-1}\lambda^{n-1}}{2}, \mathbf{z}) - (\dfrac{b^n\delta\sum_{j=1}^{n}\lambda^{j-\frac{1}{2}} + b^{n-1}\delta\sum_{j=1}^{n-1}\lambda^{j-\frac{1}{2}}}{2}, \mathbf{z}) = -(R_3^n, \mathbf{z}), & \forall\mathbf{z}\in\mathbf{W},\\
(c)(\partial_t u^n, v) - (\nabla\cdot\sigma^{n-\frac{1}{2}}, v) = (f^{n-\frac{1}{2}}, v) + (R_2^n, v), & \forall v\in L^2(\Omega),
\end{cases} \tag{3.4}$$

where

$$R_1^n = \partial_t\lambda^n - \lambda_t(t_{n-1/2}) = O(\delta^2), R_2^n = \partial_t u^n - u_t(t_{n-1/2}) = O(\delta^2),$$
$$R_3^n = \frac{\delta}{2}\sum_{j=0}^{n-1}(\lambda^j + \lambda^{j+1}) - \int_0^{t_n}\lambda(s)ds = O(\delta^2).$$

For fully discrete error estimates, we now split the errors

$$\begin{aligned}
u^n - u_h^n &= u^n - P_h u^n + P_h u^n - u_h^n = \eta^n + \varsigma^n,\\
\lambda^n - \lambda_h^n &= \lambda^n - R_h\lambda^n + R_h\lambda^n - \lambda_h^n = \rho^n + \xi^n,\\
\sigma^n - \sigma_h^n &= \sigma^n - R_h\sigma^n + R_h\sigma^n - \sigma_h^n = \gamma^n + \theta^n.
\end{aligned}$$

From (3.4),we then obtain

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Volume 2, Issue 3, May 2013

$$\begin{cases} (a)(\partial_t \xi^n, \mathbf{w}_h) + (\nabla\cdot\theta^{n-\frac{1}{2}}, \nabla\cdot\mathbf{w}_h) = -(\partial_t \rho^{n-\frac{1}{2}}, \mathbf{w}_h) + (R_1^n, \mathbf{w}_h), \ \forall \mathbf{w}_h \in \mathbf{V}_{h_\sigma}, \\ (b)(\theta^{n-\frac{1}{2}}, \mathbf{z}_h) - (\frac{a^n\xi^n + a^{n-1}\xi^{n-1}}{2}, \mathbf{z}_h) \\ -(\frac{b^n\delta\sum_{j=1}^{n}(\xi^{j-\frac{1}{2}} + \rho^{j-\frac{1}{2}}) + b^{n-1}\delta\sum_{j=1}^{n-1}(\xi^{j-\frac{1}{2}} + \rho^{j-\frac{1}{2}})}{2}, \mathbf{z}_h) \\ = -(\frac{\gamma^n + \gamma^{n-1}}{2}, \mathbf{z}_h) + (\frac{a^n\rho^n + a^{n+1}\rho^{n+1}}{2}, \mathbf{z}_h) - (R_3^n, \mathbf{z}_h), \qquad \forall \mathbf{z}_h \in \mathbf{V}_{h_\sigma}, \\ (c)(\partial_t \varsigma^n, v_h) - (\nabla\cdot(\theta^{n-\frac{1}{2}} + \gamma^{n-\frac{1}{2}}), v_h) = (R_2^n, v_h), \qquad \forall v_h \in X_{h_u}. \end{cases} \tag{3.5}$$

Lemma 3.1 *Assume that the solution of system (2.5) has regular properties that* $u_t \in L^2(H^{k+1}(\Omega))$, $\lambda_t, \sigma_t \in L^2(\mathbf{H}^{r+1}(\Omega))$. *Then we have the estimates*

$$\max_{0\le n\le M} \| \partial_t(\lambda - R_h\lambda)^n \|_{\mathbf{L}^2(\Omega)} + \max_{0\le n\le M} \| \partial_t(\sigma - R_h\sigma)^n \|_{\mathbf{L}^2(\Omega)} \le C h_\sigma^{r+1},$$
$$\max_{0\le n\le M} \| \partial_t(u - P_h u)^n \|_{L^2(\Omega)} \le C h_u^{k+1}.$$

Theorem 3.2 *Assume that* $\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t} \in L^2(H^{k+1}(\Omega))$, $\frac{\partial \lambda}{\partial t}, \frac{\partial^2 \lambda}{\partial t^2}, \frac{\partial \sigma}{\partial t}, \frac{\partial^2 \sigma}{\partial t^2} \in L^2(\mathbf{H}^{r+1}(\Omega))$, $u \in L^\infty(H^{k+1}(\Omega))$ *and* $\lambda, \sigma \in L^\infty(\mathbf{H}^{r+1}(\Omega))$, *then there exists a constant C such that*

$$(a).\max_{0\le n\le M} \| (\lambda - \lambda_h)^n \|_{\mathbf{L}^2(\Omega)} + \max_{0\le n\le M} \| (\sigma - \sigma_h)^{n-\frac{1}{2}} \|_{\mathbf{L}^2(\Omega)} \le C(h_\sigma^{r+1} + \delta^2),$$
$$(b).\max_{0\le n\le M} \| \nabla\cdot(\sigma - \sigma_h)^{n-\frac{1}{2}} \|_{L^2(\Omega)} \le C(h_\sigma^{r^*} + \delta^2),$$
$$(c).\max_{0\le n\le M} \| (u - u_h)^n \|_{L^2(\Omega)} \le C(h_u^{k+1} + h_\sigma^{r^*} + \delta^2).$$

Proof Take $\mathbf{w}_h = \theta^{n-\frac{1}{2}}$ in (3.6a) and $\mathbf{z}_h = -\partial_t \xi^n$ in (3.5a) and add the two equations to obtain

$$\begin{aligned} &\| \nabla\cdot\theta^{n-\frac{1}{2}} \|_{L^2(\Omega)}^2 + \frac{1}{2}(a^n\partial_t\xi^n, \xi^n) + \frac{1}{2}(a^{n-1}\partial_t\xi^n, \xi^{n-1}) \\ &= -(\partial_t\rho^n, \theta^{n-\frac{1}{2}}) + (R_1^n, \theta^{n-\frac{1}{2}}) + (\frac{\gamma^n + \gamma^{n-1}}{2}, \partial_t\xi^n) \\ &-(\frac{a^n\rho^n + a^{n-1}\rho^{n-1}}{2}, \partial_t\xi^n) - (\frac{b^n}{2}\frac{\delta}{2}\sum_{j=0}^{n-1}(\xi^j + \xi^{j+1} + \rho^j + \rho^{j+1}), \partial_t\xi^n) \\ &-(\frac{b^{n-1}}{2}\frac{\delta}{2}\sum_{j=0}^{n-2}(\xi^j + \xi^{j+1} + \rho^j + \rho^{j+1}), \partial_t\xi^n) + (R_3^n, \partial_t\xi^n). \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} \partial_t \| a^n\xi^n \|_{\mathbf{L}^2(\Omega)}^2 &= \frac{(a^n\xi^n, \xi^n) - (a^{n-1}\xi^{n-1}, \xi^{n-1})}{\delta} \\ &= 2(a^n\partial_t\xi^n, \xi^n) - \frac{\| (a^n)^{\frac{1}{2}}(\xi^n - \xi^{n-1}) \|_{\mathbf{L}^2(\Omega)}^2}{\delta} + (\partial_t a^n\xi^{n-1}, \xi^{n-1}). \end{aligned} \tag{3.7}$$

So, we get

$$\begin{aligned} (a^n\partial_t\xi^n, \xi^n) &= \frac{1}{2}\partial_t \| a^n\xi^n \|_{\mathbf{L}^2(\Omega)}^2 + \frac{\| (a^n)^{\frac{1}{2}}(\xi^n - \xi^{n-1}) \|_{\mathbf{L}^2(\Omega)}^2}{2\delta} - \frac{1}{2}(\partial_t a^n\xi^{n-1}, \xi^{n-1}). \\ (a^{n-1}\partial_t\xi^n, \xi^{n-1}) &= \frac{1}{2}\partial_t \| a^{n-1}\xi^n \|_{\mathbf{L}^2(\Omega)}^2 - \frac{\| (a^{n-1})^{\frac{1}{2}}(\xi^n - \xi^{n-1}) \|_{\mathbf{L}^2(\Omega)}^2}{2\delta} - \frac{1}{2}(\partial_t a^{n-1}\xi^{n-1}, \xi^{n-1}). \end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}
&(\gamma^n,\partial_t\xi^n)=\frac{(\xi^n,\gamma^n)-(\xi^{n-1},\gamma^{n-1})}{\delta}-(\partial_t\gamma^n,\xi^{n-1}),\\
&(\gamma^{n-1},\partial_t\xi^n)=\frac{(\xi^n,\gamma^{n-1})-(\xi^{n-1},\gamma^{n-2})}{\delta}-(\partial_t\gamma^{n-1},\xi^{n-1}),\\
&(a^n\rho^n,\partial_t\xi^n)=\frac{(\xi^n,a^n\rho^n)-(\xi^{n-1},a^{n-1}\rho^{n-1})}{\delta}-(\partial_t a^n\rho^n,\xi^{n-1}),\\
&(a^{n-1}\rho^{n-1},\partial_t\xi^n)=\frac{(\xi^n,a^{n-1}\rho^{n-1})-(\xi^{n-1},a^{n-2}\rho^{n-2})}{\delta}-(\partial_t a^{n-1}\rho^{n-1},\xi^{n-1}),\\
&(R_3^n,\partial_t\xi^n)=\frac{(\xi^n,R_3^n)-(\xi^{n-1},R_3^{n-1})}{\delta}-(\partial_t R_3^n,\xi^{n-1}).
\end{aligned}\tag{3.9}$$

$$\begin{aligned}
&(\frac{b^n}{2}\frac{\delta}{2}\sum_{j=0}^{n-1}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\partial_t\xi^n)=\\
&\frac{(\frac{b^n}{2}\frac{\delta}{2}\sum_{j=0}^{n-1}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^n)-(\frac{b^{n-1}}{2}\frac{\delta}{2}\sum_{j=0}^{n-2}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^{n-1})}{\delta}\\
&-\frac{1}{2}(\partial_t b^n\frac{\delta}{2}\sum_{j=0}^{n-1}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^{n-1})-\frac{(b^{n-1}(\xi^n+\xi^{n-1}+\rho^n+\rho^{n-1}),\xi^{n-1})}{4},\\
&(\frac{b^{n-1}}{2}\frac{\delta}{2}\sum_{j=0}^{n-2}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\partial_t\xi^n)=\\
&\frac{(\frac{b^{n-1}}{2}\frac{\delta}{2}\sum_{j=0}^{n-2}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^n)-(\frac{b^{n-2}}{2}\frac{\delta}{2}\sum_{j=0}^{n-3}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^{n-1})}{\delta}\\
&-\frac{1}{2}(\partial_t b^{n-1}\frac{\delta}{2}\sum_{j=0}^{n-2}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^{n-1})-\frac{(b^{n-2}(\xi^{n-1}+\xi^{n-2}+\rho^{n-1}+\rho^{n-2}),\xi^{n-1})}{4}.
\end{aligned}\tag{3.10}$$

Substitute (3.8)-(3.10) into (3.6) to get

$$\begin{aligned}
&\|\frac{\nabla\cdot(\theta^n+\theta^{n-1})}{2}\|_{L^2(\Omega)}^2+\frac{1}{4}\partial_t\|a^n\xi^n\|_{L^2(\Omega)}^2+\frac{1}{4}\partial_t\|a^{n-1}\xi^n\|_{L^2(\Omega)}^2\\
&+\frac{\|(a^n)^{\frac{1}{2}}(\xi^n-\xi^{n-1})\|_{L^2(\Omega)}^2}{4\delta}-\frac{\|(a^{n-1})^{\frac{1}{2}}(\xi^n-\xi^{n-1})\|_{L^2(\Omega)}^2}{4\delta}\\
&=(\partial_t\rho^n,\theta^{n-\frac{1}{2}})+(R_1^n,\theta^{n-\frac{1}{2}})+\frac{1}{4}(\partial_t a^n\xi^{n-1},\xi^{n-1})+\frac{1}{4}(\partial_t a^{n-1}\xi^{n-1},\xi^{n-1})\\
&+\frac{(\xi^n,\gamma^n)-(\xi^{n-1},\gamma^{n-1})}{2\delta}+\frac{(\xi^n,\gamma^{n-1})-(\xi^{n-1},\gamma^{n-2})}{2\delta}\\
&-\frac{1}{2}(\partial_t\gamma^n,\xi^{n-1})-\frac{1}{2}(\partial_t\gamma^{n-1},\xi^{n-1})+\frac{1}{2}(\partial_t a^n\rho^n,\xi^{n-1})+\frac{1}{2}(\partial_t a^{n-1}\rho^{n-1},\xi^{n-1})\\
&-\frac{(\xi^n,a^n\rho^n)-(\xi^{n-1},a^{n-1}\rho^{n-1})}{2\delta}-\frac{(\xi^n,a^{n-1}\rho^{n-1})-(\xi^{n-1},a^{n-2}\rho^{n-2})}{2\delta}\\
&-\frac{(\frac{b^n}{2}\frac{\delta}{2}\sum_{j=0}^{n-1}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^n)-(\frac{b^{n-1}}{2}\frac{\delta}{2}\sum_{j=0}^{n-2}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^{n-1})}{\delta}\\
&-\frac{(\frac{b^{n-1}}{2}\frac{\delta}{2}\sum_{j=0}^{n-2}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^n)-(\frac{b^{n-2}}{2}\frac{\delta}{2}\sum_{j=0}^{n-3}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^{n-1})}{\delta}
\end{aligned}$$

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$$+\frac{1}{2}(\partial_t b^n\frac{\delta}{2}\sum_{j=0}^{n-1}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^{n-1})+\frac{(b^{n-1}(\xi^n+\xi^{n-1}+\rho^n+\rho^{n-1}),\xi^{n-1})}{4}$$
$$+\frac{1}{2}(\partial_t b^{n-1}\frac{\delta}{2}\sum_{j=0}^{n-2}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\xi^{n-1})+\frac{(b^{n-2}(\xi^{n-1}+\xi^{n-2}+\rho^{n-1}+\rho^{n-2}),\xi^{n-1})}{4}$$
$$+\frac{(\xi^n,R_3^n)-(\xi^{n-1},R_3^{n-1})}{\delta}-(\partial_t R_3^n,\xi^{n-1}). \tag{3.11}$$

Summing from 1 to n,we find that

$$\|(a^n)^{\frac{1}{2}}\xi^n\|^2_{\mathbf{L}^2(\Omega)}+\|(a^{n-1})^{\frac{1}{2}}\xi^n\|^2_{\mathbf{L}^2(\Omega)}+4\delta\sum_{j=1}^{n}\|\nabla\cdot\theta^{j-\frac{1}{2}}\|^2_{L^2(\Omega)}$$
$$+\sum_{j=1}^{n}\|(a^j)^{\frac{1}{2}}(\xi^j-\xi^{j-1})\|^2_{\mathbf{L}^2(\Omega)}+\sum_{j=1}^{n}\|(a^{j-1})^{\frac{1}{2}}(\xi^j-\xi^{j-1})\|^2_{\mathbf{L}^2(\Omega)}$$
$$\le\|(a^0)^{\frac{1}{2}}\xi^0\|^2_{\mathbf{L}^2(\Omega)}-(\xi^0,\gamma^0)-(\xi^0,\gamma^{-1})+(\xi^0,a^0\rho^0)+(\xi^0,a^{-1}\rho^{-1})$$
$$+\varepsilon\|\xi^n\|^2_{\mathbf{L}^2(\Omega)}+C\|\rho^n\|^2_{\mathbf{L}^2(\Omega)}+C\|\rho^{n-1}\|^2_{\mathbf{L}^2(\Omega)}+C\|R_3^n\|^2_{\mathbf{L}^2(\Omega)}+C\|\gamma^n\|^2_{\mathbf{L}^2(\Omega)}$$
$$+C\|\gamma^{n-1}\|^2_{\mathbf{L}^2(\Omega)}+C\delta\sum_{j=1}^{n}\|\xi^j\|^2_{\mathbf{L}^2(\Omega)}+C\delta\sum_{j=1}^{n}\|\theta^{j-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}+C\delta\sum_{j=1}^{n}\Big[\|\partial_t\rho^j\|^2_{\mathbf{L}^2(\Omega)}$$
$$+\|\partial_t\gamma^j\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t a^j\rho^j\|^2_{\mathbf{L}^2(\Omega)}+\|\rho^j\|^2_{\mathbf{L}^2(\Omega)}+\|R_1^j\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t R_3^j\|^2_{\mathbf{L}^2(\Omega)}\Big]. \tag{3.12}$$

Choose $\mathbf{z}_h=\theta^{n-\frac{1}{2}}$ in (3.5b) to get

$$\|\theta^{n-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}\le C(\|(a^n)^{\frac{1}{2}}\xi^n\|^2_{\mathbf{L}^2(\Omega)}+\|\xi^{n-1}\|^2_{\mathbf{L}^2(\Omega)}+\|\gamma^n\|^2_{\mathbf{L}^2(\Omega)}+\|\gamma^{n-1}\|^2_{\mathbf{L}^2(\Omega)}+\|\rho^n\|^2_{\mathbf{L}^2(\Omega)}$$
$$+\|\rho^{n-1}\|^2_{\mathbf{L}^2(\Omega)})+C\|R_3^n\|^2_{\mathbf{L}^2(\Omega)}+C\delta\sum_{j=1}^{n}(\|\xi^j\|^2_{\mathbf{L}^2(\Omega)}+\|\rho^j\|^2_{\mathbf{L}^2(\Omega)}). \tag{3.13}$$

Substitute (3.13) into (3.12) and note that $\xi^0=0$ to get

$$\|\xi^n\|^2_{\mathbf{L}^2(\Omega)}+\|\theta^{n-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}+4\delta\sum_{j=1}^{n}\|\nabla\cdot\theta^{j-\frac{1}{2}}\|^2_{L^2(\Omega)}$$
$$\le C\|\rho^n\|^2_{\mathbf{L}^2(\Omega)}+C\|\rho^{n-1}\|^2_{\mathbf{L}^2(\Omega)}+C\|\gamma^n\|^2_{\mathbf{L}^2(\Omega)}+C\|\gamma^{n-1}\|^2_{\mathbf{L}^2(\Omega)}+C\|R_3^n\|^2_{\mathbf{L}^2(\Omega)}$$
$$+C\delta\sum_{j=1}^{n}\|\xi^j\|^2_{\mathbf{L}^2(\Omega)}+C\delta\sum_{j=1}^{n}\|\theta^{j-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}+C\delta\sum_{j=1}^{n}\Big[\|\partial_t\rho^j\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t\gamma^j\|^2_{\mathbf{L}^2(\Omega)}$$
$$+\|\partial_t a^j\rho^j\|^2_{\mathbf{L}^2(\Omega)}+\|\rho^j\|^2_{\mathbf{L}^2(\Omega)}+\|R_1^j\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t R_3^j\|^2_{\mathbf{L}^2(\Omega)}\Big] \tag{3.14}$$

Using Gronwall's lemma, we obtain

$$(1-C\delta)\|\xi^n\|^2_{\mathbf{L}^2(\Omega)}+(1-C\delta)\|\theta^{n-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}+4\delta\sum_{j=1}^{n}\|\nabla\cdot\theta^{j-\frac{1}{2}}\|^2_{L^2(\Omega)}$$
$$\le C\|\rho^n\|^2_{\mathbf{L}^2(\Omega)}+C\|\rho^{n-1}\|^2_{\mathbf{L}^2(\Omega)}+C\|\gamma^n\|^2_{\mathbf{L}^2(\Omega)}+C\|\gamma^{n-1}\|^2_{\mathbf{L}^2(\Omega)}+C\|R_3^n\|^2_{\mathbf{L}^2(\Omega)}$$
$$+C\delta\sum_{j=1}^{n}\Big[\|\partial_t\rho^j\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t\gamma^j\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t a^j\rho^j\|^2_{\mathbf{L}^2(\Omega)}$$
$$+\|\rho^j\|^2_{\mathbf{L}^2(\Omega)}+\|R_1^j\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t R_3^j\|^2_{\mathbf{L}^2(\Omega)}\Big] \tag{3.15}$$

Note that

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 2, Issue 3, May 2013

$$(a).\delta\sum_{j=1}^{n}\|R_1^j\|_{\mathbf{L}^2(\Omega)}^2\le C\delta^4,$$

$$(b).\delta\sum_{j=1}^{n}\|\partial_t\rho^j\|_{\mathbf{L}^2(\Omega)}^2\le C\delta\sum_{j=1}^{n}\|\frac{\partial\rho^j}{\partial t}\|_{\mathbf{L}^2(\Omega)}^2\le C\frac{T}{M}\cdot nh_\sigma^{2r+2}\le Ch_\sigma^{2r+2}, \tag{3.16}$$

$$(c).\delta\sum_{j=1}^{n}\|\partial_t\gamma^j\|_{\mathbf{L}^2(\Omega)}^2\le C\delta\sum_{j=1}^{n}\|\frac{\partial\gamma^j}{\partial t}\|_{\mathbf{L}^2(\Omega)}^2\le C\frac{T}{M}\cdot nh_\sigma^{2r+2}\le Ch_\sigma^{2r+2},$$

$$(d).\delta\sum_{j=1}^{n}\|\partial_t R_3^j\|_{\mathbf{L}^2(\Omega)}^2\le C\delta^4.$$

Therefore, substituting the above estimates into (3.15) and choosing δ_0 in such a way that for $0<\delta\le\delta_0$, $(1-C\delta)>0$, we obtain

$$\|\theta^{n-\frac{1}{2}}\|_{\mathbf{L}^2(\Omega)}^2+\|\xi^n\|_{\mathbf{L}^2(\Omega)}^2+\delta\sum_{j=1}^{n}\|\nabla\cdot\theta^{j-\frac{1}{2}}\|_{L^2(\Omega)}^2\le C(h_\sigma^{2r+2}+\delta^4). \tag{3.17}$$

By (3.5b), we obtain

$$\begin{aligned}&(\partial_t\theta^{n-\frac{1}{2}},\mathbf{z}_h)\\&=(\frac{a^n\xi^n-a^{n-2}\xi^{n-2}}{2\delta},\mathbf{z}_h)-(\frac{\gamma^n-\gamma^{n-2}}{2\delta},\mathbf{z}_h)+(\frac{a^n\rho^n-a^{n-2}\rho^{n-2}}{2\delta},\mathbf{z}_h)\\&\quad\frac{(b^n(\xi^{n-1}+\xi^n+\rho^{n-1}+\rho^n),\mathbf{z}_h)}{4}+\frac{(b^n(\xi^{n-1}+\xi^{n-2}+\rho^{n-1}+\rho^{n-2}),\mathbf{z}_h)}{4}\\&\quad(\partial_t b^{n-\frac{1}{2}}\frac{\delta}{2}\sum_{j=0}^{n-3}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\mathbf{z}_h)-(\partial_t R_3^n,\mathbf{z}_h).\end{aligned} \tag{3.18}$$

Set $\mathbf{z}_h=\partial_t\xi^{n-\frac{1}{2}}$ in (3.18) to obtain

$$\begin{aligned}&(\partial_t\theta^{n-\frac{1}{2}},\partial_t\xi^{n-\frac{1}{2}})\\&=\|(a^n)^{\frac{1}{2}}\partial_t\xi^{n-\frac{1}{2}}\|_{\mathbf{L}^2(\Omega)}^2+(a^n\partial_t\rho^{n-\frac{1}{2}},\partial_t\xi^{n-\frac{1}{2}})+(\partial_t a^{n-\frac{1}{2}}\rho^{n-2},\partial_t\xi^{n-\frac{1}{2}})\\&\quad+(\partial_t a^{n-\frac{1}{2}}\xi^{n-2},\partial_t\xi^{n-\frac{1}{2}})-(\partial_t\gamma^n,\partial_t\xi^{n-\frac{1}{2}})-(\partial_t\gamma^{n-1},\partial_t\xi^{n-\frac{1}{2}})\\&\quad+\frac{(b^n(\xi^{n-1}+\xi^n+\rho^{n-1}+\rho^n),\partial_t\xi^{n-\frac{1}{2}})}{4}+\frac{(b^n(\xi^{n-2}+\xi^{n-1}+\rho^{n-2}+\rho^{n-1}),\partial_t\xi^{n-\frac{1}{2}})}{4}\\&\quad-(\partial_t R_3^n,\partial_t\xi^{n-\frac{1}{2}})+(\partial_t b^{n-\frac{1}{2}}\frac{\delta}{2}\sum_{j=0}^{n-3}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\partial_t\xi^{n-\frac{1}{2}}).\end{aligned} \tag{3.19}$$

Transform (3.5a) that

$$(\partial_t\xi^{n-\frac{1}{2}},\mathbf{w}_h)+(\nabla\cdot\theta^{n-1},\nabla\cdot\mathbf{w}_h)=-(\partial_t\rho^{n-1},\mathbf{w}_h)+(R_1^{n-1},\mathbf{w}_h). \tag{3.20}$$

Set $\mathbf{w}_h=\partial_t\theta^{n-\frac{1}{2}}$ into (3.20) to obtain

$$\begin{aligned}&(\partial_t\xi^{n-\frac{1}{2}},\partial_t\theta^{n-\frac{1}{2}})+\frac{1}{2}\partial_t\|\nabla\cdot\theta^{n-\frac{1}{2}}\|_{L^2(\Omega)}^2-\frac{\|\nabla\cdot(\theta^n-\theta^{n-1})\|_{L^2(\Omega)}^2}{4\delta}+\frac{\|\nabla\cdot(\theta^{n-1}-\theta^{n-2})\|_{L^2(\Omega)}^2}{4\delta}\\&=-(\partial_t\rho^{n-1},\partial_t\theta^{n-\frac{1}{2}})+(R_1^{n-1},\partial_t\theta^{n-\frac{1}{2}}).\end{aligned} \tag{3.21}$$

Substitute (3.19) into (3.21) to get

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 2, Issue 3, May 2013

$$\|(a^n)^{\frac{1}{2}}\partial_t\xi^{n-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}+\frac{1}{2}\partial_t\|\nabla\cdot\theta^{n-\frac{1}{2}}\|^2_{L^2(\Omega)}-\frac{\|\nabla\cdot(\theta^n-\theta^{n-1})\|^2_{L^2(\Omega)}}{4\delta}+\frac{\|\nabla\cdot(\theta^{n-1}-\theta^{n-2})\|^2_{L^2(\Omega)}}{4\delta}$$
$$=-(\partial_t\rho^{n-1},\partial_t\theta^{n-\frac{1}{2}})+(R_1^{n-1},\partial_t\theta^{n-\frac{1}{2}})-(\partial_t a^{n-\frac{1}{2}}\xi^{n-2},\partial_t\xi^{n-\frac{1}{2}})$$
$$-(a^n\partial_t\rho^{n-\frac{1}{2}},\partial_t\xi^{n-\frac{1}{2}})-(\partial_t a^{n-\frac{1}{2}}\rho^{n-2},\partial_t\xi^{n-\frac{1}{2}})+(\partial_t\gamma^n,\partial_t\xi^{n-\frac{1}{2}})+(\partial_t\gamma^{n-1},\partial_t\xi^{n-\frac{1}{2}})$$
$$-\frac{(b^n(\xi^{n-1}+\xi^n+\rho^{n-1}+\rho^n),\partial_t\xi^{n-\frac{1}{2}})}{4}-\frac{(b^n(\xi^{n-2}+\xi^{n-1}+\rho^{n-2}+\rho^{n-1}),\partial_t\xi^{n-\frac{1}{2}})}{4}$$
$$-(\partial_t b^{n-\frac{1}{2}}\frac{\delta}{2}\sum_{j=0}^{n-3}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\partial_t\xi^{n-\frac{1}{2}})+(\partial_t R_3^n,\partial_t\xi^{n-\frac{1}{2}}). \tag{3.22}$$

Take $\mathbf{z}_h=\partial_t\theta^{n-\frac{1}{2}}$ in (3.18) to obtain

$$\|\partial_t\theta^{n-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}$$
$$=(a^n\partial_t\xi^{n-\frac{1}{2}},\partial_t\theta^{n-\frac{1}{2}})+(\partial_t a^{n-\frac{1}{2}}\xi^{n-2},\partial_t\theta^{n-\frac{1}{2}})+(a^n\partial_t\rho^{n-\frac{1}{2}},\partial_t\theta^{n-\frac{1}{2}})+(\partial_t a^{n-\frac{1}{2}}\rho^{n-2},\partial_t\theta^{n-\frac{1}{2}})$$
$$-(\partial_t\gamma^n,\partial_t\theta^{n-\frac{1}{2}})-(\partial_t\gamma^{n-1},\partial_t\theta^{n-\frac{1}{2}})+\frac{(b^n(\xi^{n-1}+\xi^n+\rho^{n-1}+\rho^n),\partial_t\theta^{n-\frac{1}{2}})}{4}$$
$$+\frac{(b^n(\xi^{n-2}+\xi^{n-1}+\rho^{n-2}+\rho^{n-1}),\partial_t\theta^{n-\frac{1}{2}})}{4}-(\partial_t R_3^n,\partial_t\theta^{n-\frac{1}{2}})$$
$$+(\partial_t b^{n-\frac{1}{2}}\frac{\delta}{2}\sum_{j=0}^{n-3}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\partial_t\theta^{n-\frac{1}{2}}). \tag{3.23}$$

Add (3.22) and (3.23) to get

$$\|(a^n)^{\frac{1}{2}}\partial_t\xi^{n-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t\theta^{n-\frac{1}{2}}\|^2_{\mathbf{L}^2(\Omega)}+\frac{1}{2}\partial_t\|\nabla\cdot\theta^{n-\frac{1}{2}}\|^2_{L^2(\Omega)}$$
$$-\frac{\|\nabla\cdot(\theta^n-\theta^{n-1})\|^2_{L^2(\Omega)}}{4\delta}+\frac{\|\nabla\cdot(\theta^{n-1}-\theta^{n-2})\|^2_{L^2(\Omega)}}{4\delta}$$
$$=-(\partial_t\rho^{n-1},\partial_t\theta^{n-\frac{1}{2}})+(R_1^{n-1},\partial_t\theta^{n-\frac{1}{2}})+(a^n\partial_t\xi^{n-\frac{1}{2}},\partial_t\theta^{n-\frac{1}{2}})$$
$$+(\partial_t a^{n-\frac{1}{2}}\xi^{n-2},\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}})+(a^n\partial_t\rho^{n-\frac{1}{2}},\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}})$$
$$+(\partial_t a^{n-\frac{1}{2}}\rho^{n-2},\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}})-(\partial_t\gamma^n,\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}})$$
$$-(\partial_t\gamma^{n-1},\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}})+\frac{(b^n(\xi^{n-1}+\xi^n+\rho^{n-1}+\rho^n),\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}})}{4}$$
$$+\frac{(b^n(\xi^{n-2}+\xi^{n-1}+\rho^{n-2}+\rho^{n-1}),\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}})}{4}-(\partial_t R_3^n,\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}})$$
$$+(\partial_t b^{n-\frac{1}{2}}\frac{\delta}{2}\sum_{j=0}^{n-3}(\xi^j+\xi^{j+1}+\rho^j+\rho^{j+1}),\partial_t\theta^{n-\frac{1}{2}}-\partial_t\xi^{n-\frac{1}{2}}). \tag{3.24}$$

Apply the Cauchy-Schwarz's inequality and the Young's inequality to obtain

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 2, Issue 3, May 2013

$$
\begin{aligned}
& a_{min} \| \partial_t \xi^{n-\frac{1}{2}} \|^2_{\mathbf{L}^2(\Omega)} + \| \partial_t \theta^{n-\frac{1}{2}} \|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2} \partial_t \| \nabla \cdot \theta^{n-\frac{1}{2}} \|^2_{L^2(\Omega)} \\
& \le \varepsilon (\| \partial_t \xi^{n-\frac{1}{2}} \|^2_{\mathbf{L}^2(\Omega)} + \| \partial_t \theta^{n-\frac{1}{2}} \|^2_{\mathbf{L}^2(\Omega)}) + C \Big[\| \xi^n \|^2_{\mathbf{L}^2(\Omega)} + \| \xi^{n-1} \|^2_{\mathbf{L}^2(\Omega)} + \| \xi^{n-2} \|^2_{\mathbf{L}^2(\Omega)} \\
& + \| \rho^n \|^2_{\mathbf{L}^2(\Omega)} + \| \rho^{n-1} \|^2_{\mathbf{L}^2(\Omega)} + \| \rho^{n-2} \|^2_{\mathbf{L}^2(\Omega)} + \| \partial_t \rho^n \|^2_{\mathbf{L}^2(\Omega)} + \| \partial_t \rho^{n-1} \|^2_{\mathbf{L}^2(\Omega)} \\
& + \| \partial_t \gamma^n \|^2_{\mathbf{L}^2(\Omega)} + \| \partial_t \gamma^{n-1} \|^2_{\mathbf{L}^2(\Omega)} + \| R_1^{n-1} \|^2_{\mathbf{L}^2(\Omega)} + \| \partial_t R_3^n \|^2_{\mathbf{L}^2(\Omega)} \Big] \\
& + C\delta^2 \sum_{j=0}^{n-2} (\| \xi^j \|^2_{\mathbf{L}^2(\Omega)} + \| \xi^{j-1} \|^2_{\mathbf{L}^2(\Omega)} + \| \rho^j \|^2_{\mathbf{L}^2(\Omega)} + \| \rho^{j-1} \|^2_{\mathbf{L}^2(\Omega)}).
\end{aligned}
$$

Using (3.16) and (3.17) and summing from 1 to n,we obtain

$$
\| \nabla \cdot \theta^{n-\frac{1}{2}} \|^2_{L^2(\Omega)} + \delta \sum_{j=1}^{n} (\| \partial_t \xi^{n-\frac{1}{2}} \|^2_{\mathbf{L}^2(\Omega)} + \| \partial_t \theta^{n-\frac{1}{2}} \|^2_{\mathbf{L}^2(\Omega)}) \le C(h_\sigma^{2r+2} + \delta^4). \tag{3.25}
$$

Choosing $v_h = \varsigma^n$ in (3.5c) and applying the Cauchy-Schwarz's inequality, the Young's inequality and (3.25),we have

$$
\begin{aligned}
& \| \varsigma^n \|^2_{L^2(\Omega)} - \| \varsigma^{n-1} \|^2_{L^2(\Omega)} + \| \varsigma^n - \varsigma^{n-1} \|^2_{L^2(\Omega)} \\
& = 2\delta (\nabla \cdot (\sigma - \sigma_h)^{n-\frac{1}{2}}, \varsigma^n) + 2\delta (R_2^n, \varsigma^n) \\
& \le C\delta \left[\| \nabla \cdot (\sigma - \sigma_h)^{n-\frac{1}{2}} \|^2_{L^2(\Omega)} + \| R_2^n \|^2_{L^2(\Omega)} + \varepsilon \| \varsigma^n \|^2_{L^2(\Omega)} \right].
\end{aligned} \tag{3.26}
$$

Summing from 1 to n and using Gronwall lemma, we obtain

$$
\| \varsigma^n \|^2_{L^2(\Omega)} \le \| \varsigma^0 \|^2_{L^2(\Omega)} + C\delta \sum_{j=1}^{n} \left[\| \nabla \cdot (\sigma - \sigma_h)^{j-\frac{1}{2}} \|^2_{L^2(\Omega)} + \| R_2^j \|^2_{L^2(\Omega)} \right]. \tag{3.27}
$$

Note that

$$
\delta \sum_{j=1}^{n} \| R_2^j \|^2_{L^2(\Omega)} \le C\delta^5 \sum_{j=1}^{n} \| \frac{\partial^3 u}{\partial t^3} \|^2_{L^2(L^2(\Omega))} \le C\delta^4 \| \frac{\partial^3 u}{\partial t^3} \|^2_{L^\infty(L^2(\Omega))} . \tag{3.28}
$$

Substituting (3.28) into (3.27) and using (3.2), (3.17) and the triangle inequality, we get

$$
\| \varsigma^n \|^2_{L^2(\Omega)} \le C(h_\sigma^{2r^*} + \delta^4). \tag{3.29}
$$

Combining (3.2), (3.3), (3.17), (3.25), (3.29) and Lemma 2, we apply the triangle inequality to complete the proof.

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