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Numerical Solution of Fourth Order Boundary Value Problems by Galerkin Method with Cubic B-splines

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Abstract—In this paper, we present a finite element method involving Galerkin method with cubic B-splines as basis functions to solve fourth order two point boundary value problems with three different cases of boundary conditions. The basis functions are redefined into a new set of basis functions which vanish on the boundary where Dirichlet type of boundary conditions and Neumann boundary conditions or second order derivative boundary conditions or mixed type boundary conditions are prescribed. The proposed method was applied to solve several examples of fourth order linear and nonlinear boundary value problems. The obtained numerical results were found to be in good agreement with exact solutions available in the literature.

***Index Terms*—Absolute Error, Cubic B-splines as Basis Functions, Fourth Order Boundary Value Problem, Galerkin Method.**

I. INTRODUCTION

In this paper, we consider a general fourth order linear boundary value problem given by

$$a_0(x)y^{(4)}(x)+a_1(x)y'''(x)+a_2(x)y''(x)+a_3(x)y'(x)+a_4(x)y(x)=b(x),\ c<x<d \tag{1}$$

subject to boundary conditions

$$y(c)=A_0,\quad y(d)=C_0,\quad y'(c)=A_1,\quad y'(d)=C_1 \tag{2a}$$

or

$$y(c)=A_0,\quad y(d)=C_0,\quad y''(c)=A_2,\quad y''(d)=C_2 \tag{2b}$$

or

$$y(c)=A_0,\quad y(d)=C_0,\quad y'(c)+\sigma_1 y(c)=A_3,\quad y'(d)+\sigma_2 y(d)=C_3 \tag{2c}$$

where $A_0, C_0, A_1, C_1, A_2, C_2, A_3, C_3, \sigma_1$ and σ_2 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$ and $b(x)$ are all continuous functions defined on the interval $[c,\ d]$.

Generally, this type of fourth order boundary value problem arises in the mathematical modeling of viscoelastic, inelastic flows, deformation of beams, arches and load bearing members like street lights and robotic arms in multi-purpose engineering systems, where elastic members serve as key elements for shedding or transmitting loads and in plate deflection theory and many other areas of engineering and applied mathematics. The existence and uniqueness of the solution for these types of problems have been discussed in Agarwal [1]. Solving such boundary value problems is analytically possible only in very rare cases. Over the years, many researchers have worked on boundary value problems by using different methods for numerical solutions. So far various numerical methods such as Finite difference method [4], Spline techniques [5]-[10], B-spline technique [11]-[13], Perturbed collocation method [14], Non polynomial spline method [15]-[23], Green element method [24], HAM [25], Variational iteration method [26]-[27], Homotopy perturbation method [29]-[30], Decomposition method [28], [30]-[31], Differential transform method [30]-[31], Sinc Galerkin method [32]-[33], Quintic B-spline collocation method [34], Galerkin method with quintic B-splines [35], Cubic B-spline collocation method [36] have been employed to solve fourth order boundary value problems. So far, fourth order boundary value problems have not been solved by using Galerkin method with cubic B-splines. This motivated us to solve a fourth order boundary value problem by Galerkin method with cubic B-splines. In this paper, we try to present a simple finite element method which involves Gelerkin approach with cubic B-splines as basis functions to solve the fourth order two point boundary value problems of the type (1)-(2). This paper is organized as follows. Section II, deals with the justification for using Galerkin Method, In Section III, a description of Galerkin method with cubic B-splines as

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basis functions is explained. In particular we first introduce the concept of cubic B-splines and followed by the proposed method with different types of boundary conditions. In Section IV, the procedure to solve the nodal parameters has been presented. In section V, the proposed method is tested on several linear and nonlinear boundary value problems. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [38]. Finally, in the last section, the conclusions are presented.

II. JUSTIFICATION FOR USING GALERKIN METHOD

In finite element method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz method, Galerkin method, Least Squares method and Collocation method etc. In Galerkin method, the residual of approximation is made orthogonal to the basis functions. When we use Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [2], [37] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient attention is given to boundary conditions [3]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed. Hence in this paper we employed the use of Galerkin method with cubic B-splines as basis functions to approximate the solution of fourth order boundary value problem.

III. DESCRIPTION OF THE METHOD

A. *Definition of cubic B-spline*

The cubic B-splines are defined in [39]-[41]. The existence of cubic spline interpolate *s(x)* to a function in a closed interval [*c*, *d*] for spaced knots (need not be evenly spaced) of a partition $c = x_0 < x_1 < \ldots < x_{n-1} < x_n = d$ is established by construction it. The construction of *s(x)* is done with the help of the cubic B-splines. Introduce six additional knots x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2} and x_{n+3} in such a way that

$$x_{-3} < x_{-2} < x_{-1} < x_0 \qquad \text{and} \qquad x_n < x_{n+1} < x_{n+2} < x_{n+3}.$$

Now the cubic B-splines $B_i(x)$'s are defined by

$$B_i(x) = \begin{cases} \sum\limits_{r=i-2}^{i+2} \dfrac{(x_r - x)_+^3}{\pi'(x_r)}, & x \in [x_{i-2}, x_{i+2}] \\ 0, & otherwise \end{cases}$$

where

$$(x_r - x)_+^3 = \begin{cases} (x_r - x), & if\ x_r \ge x \\ 0, & if\ x_r \le x \end{cases}$$

and $$\pi(x) = \prod_{r=i-2}^{i+2} (x - x_r)$$

where {$B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, …, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$}forms a basis for the space $S_3(\pi)$ of cubic polynomial splines. Schoenberg [41] has proved that cubic B-splines are the unique nonzero splines of smallest compact support with the knots at

$$x_{-3} < x_{-2} < x_{-1} < x_0 < x_1 < \ldots < x_{n-1} < x_n < x_{n+1} < x_{n+2} < x_{n+3}.$$

To solve the boundary value problem (1) subject to boundary conditions (2) by the Galerkin method with cubic B-splines as basis functions, we define the approximation for *y(x)* as

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$$y(x)=\sum_{j=-1}^{n+1}\alpha_j B_j(x) \tag{3}$$

where α_j's are the nodal parameters to be determined. In Galerkin method the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of cubic B-splines $\{B_{-1}(x), B_0(x), B_1(x), B_2(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x)\}$, the basis functions $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_{n-1}(x)$, $B_n(x)$ and $B_{n+1}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. Since we are approximating the fourth order boundary value problem by cubic B-spline polynomial, we redefine the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type boundary conditions and Neumann boundary conditions or second order derivative boundary conditions or mixed type boundary conditions are prescribed. The procedure for redefining the basis functions is as follows. Using the definition of cubic B-splines and the Dirichlet boundary condition of (2), we get the approximate solution at the boundary points as

$$A_0=y(c)=y(x_0)=\alpha_{-1}B_{-1}(x_0)+\alpha_0 B_0(x_0)+\alpha_1 B_1(x_0) \tag{4}$$

$$C_0=y(d)=y(x_n)=\alpha_{n-1}B_{n-1}(x_n)+\alpha_n B_n(x_n)+\alpha_{n+1}B_{n+1}(x_n) \tag{5}$$

Eliminating α_{-1} and α_{n+1} from the equations (3), (4) and (5), we get

$$y(x)=w_1(x)+\sum_{j=0}^{n}\alpha_j P_j(x) \tag{6}$$

where
$$w_1(x)=\frac{A_0}{B_{-1}(x_0)}B_{-1}(x)+\frac{C_0}{B_{n+1}(x_n)}B_{n+1}(x) \tag{7}$$

and
$$P_j(x)=\begin{cases} B_j(x)-\dfrac{B_j(x_0)}{B_{-1}(x_0)}B_{-1}(x), & j=0,1\\ B_j(x), & j=2,3,\ldots,n-2\\ B_j(x)-\dfrac{B_j(x_n)}{B_{n+1}(x_n)}B_{n+1}(x), & j=n-1,n. \end{cases} \tag{8}$$

B. Redefinition of basis functions with boundary conditions (2a)

Using the Neumann boundary conditions of (2a) to the approximate solution $y(x)$ in (6), we get

$$A_1=y'(c)=y'(x_0)=w_1'(x_0)+\alpha_0 P_0'(x_0)+\alpha_1 P_1'(x_0) \tag{9}$$

$$C_1=y'(d)=y'(x_n)=w_1'(x_n)+\alpha_{n-1}P_{n-1}'(x_n)+\alpha_n P_n'(x_n) \tag{10}$$

Eliminating α_0 and α_n from the equations (9), (10) and (6), we get approximation for $y(x)$ as

$$y(x)=w(x)+\sum_{j=1}^{n-1}\alpha_j \tilde{B}_j(x) \tag{11}$$

where
$$w(x)=w_1(x)+\frac{A_1-w_1'(x_0)}{P_0'(x_0)}P_0(x)+\frac{C_1-w_1'(x_n)}{P_n'(x_n)}P_n(x) \tag{12}$$

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and $$\tilde{B}_j(x) = \begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_0'(x_0)} P_0(x), & j = 1 \\ P_j(x), & j = 2,3,\dots,n-2 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_n'(x_n)} P_n(x), & j = n-1. \end{cases} \quad (13)$$

C. Redefinition of basis functions with boundary conditions (2b)

Using the second order derivative boundary conditions of (2b) to the approximate solution *y(x)* in (6), we get

$$A_2 = y''(c) = y''(x_0) = w_1''(x_0) + \alpha_0 P_0''(x_0) + \alpha_1 P_1''(x_0) \quad (14)$$

$$C_2 = y''(d) = y''(x_n) = w_1''(x_n) + \alpha_{n-1} P_{n-1}''(x_n) + \alpha_n P_n''(x_n) \quad (15)$$

Eliminating α_0 and α_n from the equations (14), (15) and (6), we get approximation for *y(x)* as

$$y(x) = w(x) + \sum_{j=1}^{n-1} \alpha_j \tilde{B}_j(x) \quad (16)$$

where $$w(x) = w_1(x) + \frac{A_1 - w_1''(x_0)}{P_0''(x_0)} P_0(x) + \frac{C_1 - w_1''(x_n)}{P_n''(x_n)} P_n(x) \quad (17)$$

and $$\tilde{B}_j(x) = \begin{cases} P_j(x) - \dfrac{P_j''(x_0)}{P_0''(x_0)} P_0(x), & j = 1 \\ P_j(x), & j = 2,3,\dots,n-2 \\ P_j(x) - \dfrac{P_j''(x_n)}{P_n''(x_n)} P_n(x), & j = n-1. \end{cases} \quad (18)$$

D. Redefinition of basis functions with boundary conditions (2c)

Using the mixed boundary conditions of (2c) to the approximate solution *y(x)* in (6), we get,

$$A_3 = y'(c) + \sigma_1 y(c) = y'(x_0) + \sigma_1 y(x_0) = w_1'(x_0) + \alpha_0 P_0'(x_0) + \alpha_1 P_1'(x_0) + \sigma_1 w_1(x_0) \quad (19)$$

$$\begin{aligned} C_3 &= y'(d) + \sigma_2 y(d) = y'(x_n) + \sigma_2 y(x_n) \\ &= w_1'(x_n) + \alpha_{n-1} P_{n-1}'(x_n) + \alpha_n P_n'(x_n) + \sigma_2 w_1(x_n) \end{aligned} \quad (20)$$

Eliminating α_0 and α_n from the equations (19), (20) and (6), we get approximation for *y(x)* as

$$y(x) = w(x) + \sum_{j=1}^{n-1} \alpha_j \tilde{B}_j(x) \quad (21)$$

where

$$w(x) = w_1(x) + \frac{A_3 - w_1'(x_0) - \sigma_1 w_1(x_0)}{P_0'(x_0)} P_0(x) + \frac{C_3 - w_1'(x_n) - \sigma_2 w_1(x_n)}{P_n'(x_n)} P_n(x) \quad (22)$$

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and $$\widetilde{B}_j(x)=\begin{cases} P_j(x)-\dfrac{P_j'(x_0)}{P_0'(x_0)}P_0(x), & j=1 \\ P_j(x), & j=2,3,\ldots,n-2 \\ P_j(x)-\dfrac{P_j'(x_n)}{P_n'(x_n)}P_n(x), & j=n-1. \end{cases} \quad (23)$$

Now the new set of basis functions for the approximation *y(x)* is { $\widetilde{B}_j(x)$, j=1,2,…,n-1} as described in (13) or (18) or (23). Applying the Galerkin method to (1) with the new set of basis functions, we get

$$\int_{x_0}^{x_n}[a_0(x)y^{(4)}(x)+a_1(x)y'''(x)+a_2(x)y''(x)+a_3(x)y'(x)+a_4(x)y(x)]\widetilde{B}_i(x)\,dx$$
$$=\int_{x_0}^{x_n}b(x)\widetilde{B}_i(x)\,dx \qquad for \;\; i=1,2,3,\ldots,n-1. \quad (24)$$

E. Method with boundary conditions (2a)

Integrating by parts the first two terms on the left hand side of (24) and after applying the boundary conditions prescribed in (2a), we get

$$\int_{x_0}^{x_n}a_0(x)y^{(4)}(x)\widetilde{B}_i(x)\,dx=\int_{x_0}^{x_n}\frac{d^2}{dx^2}[a_0(x)\widetilde{B}_i(x)]\,y''(x)\,dx \quad (25)$$

$$\int_{x_0}^{x_n}a_1(x)y'''(x)\widetilde{B}_i(x)\,dx=-\int_{x_0}^{x_n}\frac{d}{dx}[a_1(x)\widetilde{B}_i(x)]\,y''(x)\,dx \quad (26)$$

Substituting (25), (26) in (24) and using the approximation for *y(x)* given in (11) and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$A\alpha=B \quad (27)$$

where $A=[a_{ij}]$;

$$a_{ij}=\int_{x_0}^{x_n}\{[\frac{d^2}{dx^2}(a_0(x)\widetilde{B}_i(x))-\frac{d}{dx}(a_1(x)\widetilde{B}_i(x))+a_2(x)\widetilde{B}_i(x)]\widetilde{B}_j''(x)+a_3(x)\widetilde{B}_i(x)\widetilde{B}_j'(x)$$
$$+a_4(x)\widetilde{B}_i(x)\widetilde{B}_j(x)\}\,dx \qquad for \;\; i=1,2,\ldots,n-1;\; j=1,2,\ldots,n-1 \quad (28)$$

$B=[b_i]$;

$$b_i=\int_{x_0}^{x_n}\{b(x)\widetilde{B}_i(x)-[\frac{d^2}{dx^2}(a_0(x)\widetilde{B}_i(x))-\frac{d}{dx}(a_1(x)\widetilde{B}_i(x))+a_2(x)\widetilde{B}_i(x)]w''(x)$$
$$-a_3(x)\widetilde{B}_i(x)w'(x)-a_4(x)\widetilde{B}_i(x)w(x)\}\,dx \qquad for \;\; i=1,2,3,\ldots,n-1 \quad (29)$$

and $\alpha=[\alpha_1\;\alpha_2\ldots\alpha_{n-1}]^T$.

F. Method with boundary conditions (2b)

Integrating by parts the first two terms on the left hand side of (24) and after applying the boundary conditions prescribed in (2b), we get

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$$\int_{x_0}^{x_n} a_0(x)y^{(4)}(x)\tilde{B}_i(x)\,dx = -[\frac{d}{dx}(a_0(x)\tilde{B}_i(x))\,y''(x)]_{x_0}^{x_n} - \int_{x_0}^{x_n} \frac{d^3}{dx^3}(a_0(x)\tilde{B}_i(x))\,y'(x)\,dx \tag{30}$$

$$\int_{x_0}^{x_n} a_1(x)y'''(x)\tilde{B}_i(x)\,dx = -\int_{x_0}^{x_n} \frac{d}{dx}[a_1(x)\tilde{B}_i(x)]\,y''(x)\,dx \tag{31}$$

Substituting (30), (31) in (24) and using the approximation for *y(x)* as in (16) and after rearranging the terms for the resulting equations, we get a system of equations in the matrix form as

$$A\alpha = B \tag{32}$$

where $A = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \{[a_2(x)\tilde{B}_i(x) - \frac{d}{dx}(a_1(x)\tilde{B}_i(x))]\tilde{B}_j''x) + [a_3(x)\tilde{B}_i(x) - \frac{d^3}{dx^3}(a_0(x)\tilde{B}_i(x))]\tilde{B}_j'(x)$$
$$+ a_4(x)\tilde{B}_i(x)\tilde{B}_j(x)\}\,dx \qquad \textit{for} \;\; i = 1,2,\ldots,n-1;\; j = 1,2,\ldots,n-1 \tag{33}$$

$B = [b_i]$;

$$b_i = \int_{x_0}^{x_n} \{b(x)\tilde{B}_i(x) + [-a_2(x)\tilde{B}_i(x) + \frac{d}{dx}(a_1(x)\tilde{B}_i(x))]w''(x) + w'(x)[\frac{d^3}{dx^3}(a_0(x)\tilde{B}_i(x))$$
$$- a_3(x)\tilde{B}_i(x)] - a_4(x)\tilde{B}_i(x)w(x)\}\,dx + C_2[\frac{d}{dx}(a_0(x)\tilde{B}_i(x))]_{x_n} - A_2[\frac{d}{dx}(a_0(x)\tilde{B}_i(x))]_{x_0}$$

for i=1,2,…,n-1 (34)

and $\alpha = [\alpha_1\; \alpha_2 \ldots \alpha_{n-1}]^T$.

G. *Method with boundary conditions (2c)*

Integrating by parts the first two terms on the left hand side of (21) and after applying the boundary conditions prescribed in (2c), we get

$$\int_{x_0}^{x_n} a_0(x)y^{(4)}(x)\tilde{B}_i(x)\,dx = \int_{x_0}^{x_n} \frac{d^2}{dx^2}[a_0(x)\tilde{B}_j(x)]\,y''(x)\,dx - [\frac{d}{dx}(a_0(x)\tilde{B}_i(x))y''(x)]_{x_0}^{x_n} \tag{35}$$

$$\int_{x_0}^{x_n} a_1(x)y''(x)\tilde{B}_i(x)\,dx = -\int_{x_0}^{x_n} \frac{d}{dx}[a_1(x)\tilde{B}_i(x)]\,y''(x)\,dx \tag{36}$$

Substituting (35), (36) in (24) and using the approximation for *y(x)* as in (16) and after rearranging the terms for the resulting equations, we get a system of equations in the matrix form as

$$A\alpha = B \tag{37}$$

where $A = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \{[\frac{d^2}{dx^2}(a_0(x)\tilde{B}_i(x)) - \frac{d}{dx}(a_1(x)\tilde{B}_i(x)) + a_2(x)\tilde{B}_i(x)]\tilde{B}_j''(x) + a_3(x)\tilde{B}_i(x)\tilde{B}_j'(x)$$
$$+ a_4(x)\tilde{B}_i(x)\tilde{B}_j(x)\}\,dx - [\frac{d}{dx}(a_0(x)\tilde{B}_i(x))\,\tilde{B}_j''(x)]_{x_n} + [\frac{d}{dx}(a_0(x)\tilde{B}_i(x))\,\tilde{B}_j''(x)]_{x_0}$$

$$\text{for } i = 1,2,3,\ldots,n-1;\;\; j = 1,2,3,\ldots,n-1 \tag{38}$$

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$B = [b_i];$

$$b_i = \int_{x_0}^{x_n} \{b(x)\tilde{B}_i(x) + [-\frac{d^2}{dx^2}(a_0(x)\tilde{B}_i(x)) + \frac{d}{dx}(a_1(x)\tilde{B}_i(x)) - a_2(x)\tilde{B}_i(x)]w''(x)$$

$$- a_3(x)\tilde{B}_i(x)w'(x) - a_4(x)\tilde{B}_i(x)w(x)\}\,dx + [\frac{d}{dx}(a_0(x)\tilde{B}_i(x))\,w''(x)]_{x_n} \quad (39)$$

$$-[\frac{d}{dx}(a_0(x)\tilde{B}_i(x))\,w''(x)]_{x_0} \quad for \quad i = 1,2,3,\ldots,n-1$$

and $\alpha = [\alpha_1\ \alpha_2 \ldots \alpha_{n-1}]^T$.

IV. PROCEDURE TO FIND SOLUTION FOR NODAL PARAMETERS

A typical integral element in the matrix A is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} r_i(x)\,r_j(x)\,Z(x)\,dx$ and $r_i(x), r_j(x)$ are the cubic B-spline basis functions or their derivatives. It may be noted that $I_m = 0$ if $(x_{i-2}, x_{i+2}) \cap (x_{j-2}, x_{j+2}) \cap (x_m, x_{m+1}) = \phi$. To evaluate each I_m, we employed 4-point Gauss-Legendre quadrature formula. Thus the stiff matrix A is a seven diagonal band matrix. The nodal parameter vector α has been obtained from the system $A\alpha = B$ using a band matrix solution package. We have used FORTRAN-90 program to solve the boundary value problems (1) and (2) by the proposed method.

V. NUMERICAL RESULTS

To demonstrate the applicability of the proposed method for solving the fourth order boundary value problems of the type (1) and (2), we considered four linear boundary value problems and four nonlinear boundary value problems. Numerical results obtained for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1: Consider the linear boundary value problem

$$y^{(4)} + 4y = 1, \qquad -1 < x < 1 \quad (40)$$

subject to $y(-1) = y(1) = 0,\ y'(-1) = y'(1) = \dfrac{\sinh 2 - \sin 2}{4(\cosh 2 + \cos 2)}$.

The exact solution for the above problem is

$$y = 0.25[1 - 2\frac{\sinh 1 \sin 1 \sinh x \sin x + \cosh 1 \cos 1 \cosh x \cos x}{\cos 2 + \cosh 2}].$$

The proposed method is tested on this problem where the domain [-1, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is 4.470348 x 10^{-7}.

Example 2: Consider the linear boundary value problem

$$y^{(4)} + xy = -(8 + 7x + x^3)e^x, \qquad 0 < x < 1 \quad (41)$$

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subject to $y(0) = y(1) = 0,\ y'(0) = y'(1) = -e$.

The exact solution for the above problem is $y = x(1-x)e^{x}$. The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is 1.192093 x 10^{-6}.

Example 3: Consider the linear boundary value problem

$$y^{(4)} - y = -4(2x\cos x + 3\sin x), \qquad 0 < x < 1 \tag{42}$$

subject to $y(0) = 0,\ y''(0) = 0, y(1) = 0,\ y''(1) = 2\sin 1 + 4\cos 1$.

The exact solution for the above problem is $y = (x^2 - 1)\sin x$. The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3. The maximum absolute error obtained by the proposed method is 5.0066790 x 10^{-6}.

Table 1: Numerical results for Example 1

x	Exact Solution	Absolute error by proposed method
-0.8	3.976926E-02	3.352761E-08
-0.6	7.498498E-02	5.215406E-08
-0.4	1.023106E-01	1.788139E-07
-0.2	1.195382E-01	3.278255E-07
0.0	1.254157E-01	4.470348E-07
0.2	1.195382E-01	4.172325E-07
0.4	1.023106E-01	3.129244E-07
0.6	7.498498E-02	1.490116E-07
0.8	3.976926E-02	3.352761E-08

Table 2: Numerical results for Example 2

x	Exact Solution	Absolute error by proposed method
0.1	9.946539E-02	1.937151E-07
0.2	1.954244E-01	5.513430E-07
0.3	2.834704E-01	9.238720E-07
0.4	3.580379E-01	1.192093E-06
0.5	4.121803E-01	1.192093E-06
0.6	4.373085E-01	1.072884E-06
0.7	4.228888E-01	9.536743E-07
0.8	3.560865E-01	5.066395E-07
0.9	2.213642E-01	8.940697E-08

Table 3: Numerical results for Example 3

x	Exact Solution	Absolute error by proposed method
0.1	-9.883508E-02	1.519918E-06
0.2	-1.907226E-01	2.905726E-06
0.3	-2.689234E-01	4.053116E-06
0.4	-3.271114E-01	4.917383E-06
0.5	-3.595692E-01	5.006790E-06
0.6	-3.613712E-01	4.500151E-06
0.7	-3.285510E-01	3.755093E-06
0.8	-2.582482E-01	2.622604E-06
0.9	-1.488321E-01	1.311302E-06

Table 4: Numerical results for Example 4

x	Exact Solution	Absolute error by proposed method
0.1	1.105171	2.145767E-06
0.2	1.221403	6.675720E-06
0.3	1.349859	1.168251E-05
0.4	1.491825	1.585484E-05
0.5	1.648721	1.692772E-05

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0.6	1.822119	1.561642E-05
0.7	2.013753	1.239777E-05
0.8	2.225541	7.152557E-06
0.9	2.459603	2.145767E-06

Example 4: Consider the linear boundary value problem

$$y^{(4)} - y = 0, \qquad 0 < x < 1 \tag{43}$$

subject to $y(0)=1,\quad y(1)=-e\,,\quad y'(0)-y(0)=0,\; y'(1)-y(1)=0.$

The exact solution for the above problem is $y=(x^2-1)\sin x$. The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 4. The maximum absolute error obtained by the proposed method is 1.692772 x 10^{-5}.

Example 5: Consider the nonlinear boundary value problem

$$y^{(4)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48, \qquad 0 < x < 1 \tag{44}$$

subject to $y(0) = y'(0) = 0,\quad y(1) = y'(1) = 1$.

The exact solution for the above problem is $y = x^5 - 2x^4 + 2x^2$. The nonlinear boundary value problem (44) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [38] as

$$y^{(4)}_{(n+1)} - [2y_{(n)}]\,y_{(n+1)} = -x^{10} + 4\mathrm{x}^9 - 4\mathrm{x}^8 - 4\mathrm{x}^7 + 8\mathrm{x}^6 - 4\mathrm{x}^4 + 120x - 48 - [y_{(n)}]^2$$

$$\text{n= }0,1,2,3,\ldots \tag{45}$$

subject to $y_{(n+1)}(0) = y'_{(n+1)}(0) = 0,\quad y_{(n+1)}(1) = y'_{(n+1)}(1) = 1$.

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for *y*. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (45). The obtained numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is 8.834465 x 10^{-7}.

Example 6: Consider the nonlinear boundary value problem

$$y^{(4)} = \sin x + \sin^2 x - [y'']^2\,, \qquad 0 < x < 1 \tag{46}$$

Subject to $y(0) = y'(0) = 1,\quad y(1) = \sin 1,\quad y'(1) = \cos 1$.

The exact solution for the above problem is $y = \sin x$. The nonlinear boundary value problem (46) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [38] as

$$y^{(4)}_{(n+1)} + [2y_{(n)}]\,y_{(n+1)} = \sin x + \sin^2 x + [y_{(n)}]^2\,,\quad n = 0,1,2,3,\ldots \tag{47}$$

subject to $y_{(n+1)}(0) = y'_{(n+1)}(0) = 1,\quad y_{(n+1)}(1) = \sin 1,\quad y'_{(n+1)}(1) = \cos 1$.

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for *y*. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (47). The obtained numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is 1.847744 x 10^{-6}.

Example 7: Consider the nonlinear boundary value problem

$$y^{(4)} - 6e^{-4y} = -12\,(1+x)^{-4}\,, \qquad 0 < x < 1 \tag{48}$$

subject to $y(0) = 0,\quad y'(0) = 1,\quad y(1) = \ln 2,\quad y'(1) = 0.5$.

The exact solution for the above problem is $y = \ln(1 + x)$. The nonlinear boundary value problem (48) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [38] as

$$y^{(4)}_{(n+1)} + [24e^{-4y_{(n)}}]\,y_{(n+1)} = -12(1+x)^{-4} + e^{-4y_{(n)}}(6 + 24y_{(n)}),\quad n = 0,1,2,\ldots \tag{49}$$

subject to $y_{(n+1)}(0) = 0,\quad y'_{(n+1)}(0) = 1,\quad y_{(n+1)}(1) = \ln 2,\quad y'_{(n+1)}(1) = 0.5$.

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Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for *y*. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (49). The obtained numerical results for this problem are presented in Table 7. The maximum absolute error obtained by the proposed method is 5.245209 x 10^{-6}.

Example 8: Consider the nonlinear boundary value problem

$$y^{(4)}+\frac{x^2}{1+y^2}=-72(1-5x+5x^2)+\frac{x^2}{1+(x-x^2)^6}\,,\qquad 0<x<1 \tag{50}$$

subject to $y(0)=0,\ y'(0)=0,\ y(1)=0,\ y'(1)=0$.

The exact solution for the above problem is $y=x^3(1-x)^3$. The nonlinear boundary value problem (50) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [38] as

$$y_{(n+1)}^{(4)}-\frac{2x^2y_{(n)}}{(1+[y_{(n)}]^2)^2}y_{(n+1)}=\frac{x^2}{1+(x-x^2)^6}-72(1-5x+5x^2)-\frac{2x^2[y_{(n)}]^2}{(1+[y_{(n)}]^2)^2}-\frac{x^2}{1+[y_{(n)}]^2}$$

n= 0,1,2,3,… (51)

subject to $y_{(n+1)}(0)=0,\ y'_{(n+1)}(0)=0,\ y_{(n+1)}(1)=0,\ y'_{(n+1)}(1)=0$.

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for *y*. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (51). The obtained numerical results for this problem are presented in Table 8. The maximum absolute error obtained by the proposed method is 1.005828 x 10^{-7}.

Table 5: Numerical results for Example 5

x	Exact Solution	Absolute error by proposed method
0.1	1.981000E-02	6.146729E-08
0.2	7.712000E-02	1.937151E-07
0.3	1.662300E-01	3.576279E-07
0.4	2.790400E-01	5.066395E-07
0.5	4.062500E-01	7.450581E-07
0.6	5.285600E-01	8.344650E-07
0.7	6.678700E-01	4.172325E-07
0.8	7.884800E-01	1.192093E-07
0.9	8.982900E-01	5.960464E-08

Table 6: Numerical results for Example 6

x	Exact Solution	Absolute error by proposed method
0.1	9.983342E-02	3.725290E-08
0.2	1.986693E-01	2.384186E-07
0.3	2.955202E-01	6.854534E-07
0.4	3.894183E-01	1.162291E-06
0.5	4.794255E-01	1.758337E-06
0.6	1.758337E-06	1.847744E-06
0.7	6.442177E-01	1.311302E-06
0.8	7.173561E-01	6.556511E-07
0.9	7.833269E-01	1.192093E-07

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Table 7: Numerical results for Example 7

x	Exact Solution	Absolute error by proposed method
0.1	9.531018E-02	5.885959E-07
0.2	1.823216E-01	1.892447E-06
0.3	2.623643E-01	3.427267E-06
0.4	3.364722E-01	4.708767E-06
0.5	4.054651E-01	5.245209E-06
0.6	4.700036E-01	4.976988E-06
0.7	5.306283E-01	4.053116E-06
0.8	5.877867E-01	2.443790E-06
0.9	6.418539E-01	8.344650E-07

Table 8: Numerical results for Example 8

x	Exact Solution	Absolute error by proposed method
0.1	7.290000E-04	1.169974E-08
0.2	4.096000E-03	3.771856E-08
0.3	9.261001E-03	6.705523E-08
0.4	1.382400E-02	8.847564E-08
0.5	1.562500E-02	1.005828E-07
0.6	1.382400E-02	9.126961E-08
0.7	9.261001E-03	6.798655E-08
0.8	4.096000E-03	3.725290E-08
0.9	7.289993E-04	1.082662E-08

VI. Conclusion

In this paper, we have deployed a Galerkin method with cubic B-splines as basis functions to solve fourth order boundary value problems with three different cases of boundary conditions. The cubic B-spline basis set has been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet boundary conditions and Neumann boundary conditions or second order derivative boundary conditions or mixed boundary conditions are prescribed. The proposed method has been applied to solve several linear and nonlinear fourth order boundary value problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The strength of the proposed method lies in its easy applicability, accurate and efficient to solve fourth order boundary value problems.

References

[1] R. P. Agarwal, "Boundary value problems for higher order differential equations," World Scientific, Singapore, 1986.

[2] L. Bers, F. John and M. Schecheter, "Partial differential equations, John Wiley Inter Science," New York, 1964.

[3] A. R. Mitchel and R. Wait, "The finite element method in partial differential Equations," John Wiley and Sons, London, 1977.

[4] J. Rashidinia, "Finite difference methods for a class of two point boundary value problems," IUST International Journal of Engineering Science, vol. 19, pp. 67-72, 2008.

[5] D. J. Fyfe, "The use of cubic splines in the solution of certain fourth order boundary value problems," The Computer Journal, vol. 13, pp. 204-205, May. 1970.

[6] N. Papamichael and A. J. Worsey, "A cubic spline method for the solution a linear fourth order two point boundary value problem," Journal of Computational and Applied Mathematics, vol. 7, pp. 187-189, 1981.

[7] S. A. Warsi and R. Usmani, "Smooth spline solutions for boundary value problems in plate deflection theory," Computers and Mathematics with Applications, vol. 6, pp. 205-211, 1980.

[8] Shahid S. Siddiqi and Ghazala Akram, "Quintic spline solutions of fourth order boundary value problems," International Journal of Numerical Analysis and Modeling, vol. 5, pp. 101-111, 2006.

[9] J. Rashidinia and R. Mohammadi and R. Jalilian, "Quintic spline solution of boundary value problems in the plate deflection theory," Computer Science Engineering and Electrical Engineering, vol. 16, pp. 53-59, June 2009.

[10] R. A. Usmani, "Smooth spline approximations for the solution of a boundary value problem with engineering applications," Journal of Computational and Applied Mathematics, vol. 6, pp. 93-98, 1980.

[11] M. Sakai and R. Usmani, "Spline solutions for nonlinear fourth order two point boundary value problems," Publications of the Research Institute for Mathematical Sciences, Kyoto University, vol. 19, pp. 135-144, 1983.

[12] Yogesh Gupta and P. K. Srivastava, "A computational method for solving two point boundary value problems of order four," International Journal of Computer Technology and Applications, vol. 2, pp. 1426-1431, Oct. 2011.

[13] Yogesh Gupta and Manoj Kumar, "B-spline method for solution of linear fourth order boundary value problem," Canadian Journal on Computing in Mathematics, Natural Sciences, Engineering and Medicine, vol. 2, pp. 166-169, Aug. 2011.

[14] O. A. Taiwo, "Application of collocation approximations for the solution of fourth order nonlinear boundary value problems," Proceedings of the Pakistan Academy of Sciences, vol. 2, pp. 85-89, 2009.

[15] Shahid S. Siddiqi and Ghazala Akram, "Numerical solution of system of fourth order boundary value problems using cubic non-polynomial spline method," Applied Mathematics and Computation, vol. 190, pp. 652-661, 2007.

[16] J. Rashidinia and F. Barati, "Non-polynomial quartic spline solution of boundary value problem," International Journal of Mathematical Modeling and Computations, vol. 1, pp. 35-44, Jan. 2011.

[17] F. A. Abul EI-Salam and Z. A. Zaki, "The numerical solution of linear fourth order boundary value problems using nonpoynomial spline technique," Journal of American Science, vol. 6, pp. 310-316, 2010.

[18] M. A. Ramadan, I. F. Lashien et al., "Quintic nonpolynomial spline solutions for fourth order two boundary value problem," Nonlinear Science and Numerical Simulation, vol. 14, pp. 1105-1114, 2009.

[19] P. K. Srivastava, M. Kumar et al., "Solution of fourth order nonlinear boundary value problems by numerical algorithm based on non-polynomial quintic splines," Journal of Numerical Methods and Stochastic, vol. 1, pp. 13-25, 2012.

[20] R. Jalilian, "Non-polynomial spline solutions for special nonlinear fourth order boundary value problems," International Journal of Mathematical Modeling and Computations, vol. 1, pp. 135-147, 2011.

[21] O. A. Taiwo and O. M. Ogunlaram, "A non-polynomial method for solving linear fourth order boundary value problems," International Journal of the Physical Sciences, vol. 6, pp. 3246-3254, 2011.

[22] Siraj-ul-Islam, Ikram A. Tirmizi et al., "A class of methods based on non-polynomial spline functions for the solution of a special fourth-order boundary-value problems with engineering applications," Applied Mathematics and Computation, vol. 174, pp. 1169-1180, 2006.

[23] Arshad Khan, "Non-polynomial septic splines approach to the solution of fourth order two point boundary value problems," International Journal of Nonlinear Science, vol. 13, pp. 363-372, 2012.

[24] Okey Oseloka Onyejekwe, "A green element method for fourth order ordinary differential equations," Advances in Engineering Software, vol. 25, pp. 517-525, 2004.

[25] Songxin Liang and David J. Jeffrey, "An efficient analytical approach for solving fourth order boundary value problems," Computer Physics Communications, vol. 180, pp. 2034-2040, June 2009.

[26] Muhammad Aslam Noor and Syed Tauseef Mohyud-Din, "An efficient method for fourth-order boundary value problems," Computers and Mathematics with Applications, vol. 54, pp. 1101-1111, 2007.

[27] Lan Xu, "The variational iteration method for fourth order boundary value problems with engineering applications," Chaos, Solitons and Fractals, vol. 39, pp. 1386-1394, 2009.

[28] Waleed Al-Hayani and Luis Casasus, "Approximate analytical solution of fourth order boundary value problems," Numerical Algorithms, vol. 13, pp. 67-78, 2005.

[29] Syed Tauseef Mohyud-Din and Muhammad Aslam Noor, "Homotopy perturbation method for solving fourth- order boundary value problems," Mathematical Problems in Engineering, Article ID: 98602, 2007.

[30] Shaher Momani and Muhammad Aslam Noor, "Numerical comparison of methods for solving a special fourth order boundary value problem," Applied Mathematics and Computation, vol. 191, pp. 218–224, 2007.

[31] Vedat Suat Erturk and Shaher Momani, "Comparing numerical methods for solving fourth order boundary value problems," Applied Mathematics and Computation, vol. 188, pp. 1963-1968, 2007.

ISSN: 2319-5967
ISO 9001:2008 Certified
International Journal of Engineering Science and Innovative Technology (IJESIT)
Volume 2, Issue 3, May 2013

[32] Mohamed El-Gamel, S. H. Behiry et al., "Numerical method for the solution of special nonlinear fourth-order boundary value problems," Applied Mathematics and Computation, vol.145, pp. 717-734, 2003.

[33] Ralph C. Smith, Gary A. Bogar et al., "The Sinc-Galerkin method for fourth order differential equations," SIAM Journal on Numerical Analysis, vol. 28, pp. 760-788, 1991.

[34] K. N. S. Kasi Viswanadham and P. Murali Krishna, "Quintic B-splines collocation method for fourth order boundary value problems," Proceedings of 54th Congress of Indian Society of Theoretical and Applied Mechanics (An International meet), pp. 235-243, 2010.

[35] K. N. S. Kasi Viswanadham and P. Murali Krishna, "Numerical solutions of fourth order boundary value problems by galerkin method with quintic B-splines," International Journal of Nonlinear Science, vol. 10, pp. 222-230, 2010.

[36] K. N. S. Kasi Viswanadham and Y. Showri Raju, "Cubic B-spline collocation method for fourth order boundary value problems," International Journal of Nonlinear Science, vol. 14, pp. 336-344, 2012.

[37] J. L. Lions and E. Magenes, "Non-Homogeneous boundary value problem and applications," Springer-Verlag, Berlin, 1972.

[38] R. E. Bellman and R. E. Kalaba, "Quasilinearzation and nonlinear boundary value problems," American Elsevier, New York, 1965.

[39] P. M. Prenter, "Splines and variational methods," John-Wiley and Sons, New York, 1989.

[40] Carl de-Boor, "A practical guide to splines," Springer-Verlag, 1978.

[41] I. J. Schoenberg, "On spline functions," MRC Report 625, University of Wisconsin, 1966.

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