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Some Applications of Quadrature Methods

S. B. Sahoo, M. Acharya, B. P. Acharya
Institute of Technical Education and Research, S'O'A University, Bhubaneswar (India)

Abstract: The Newton-Raphson (N-R) method for the solution of transcendental equations and optimization of unimodal functions has been modified by making use of the generalized Cauchy integral formula and the transformed Gauss-Legendre quadrature rules. The modified method has been applied for root finding and also the optimization of functions.

AMS Classification: 65 D 25, 65 D 30

***Keywords*: Gauss-Legendre Rules, Generalized Cauchy Integral Formula, Newton-Raphson Method.**

I. INTRODUCTION

Finding out the approximate solution of transcendental and polynomial equations arises quite often in scientific investigations and engineering (cf. Aitkinson[1]). Similarly finding out the optimum of unimodal functions in an interval of uncertainty has several applications in engineering and economics (cf. Deb[2]). Therefore a number of numerical methods have been devised for root finding and optimization of functions. Amongst these methods the N-R method is a familiar one. The N-R method involves derivatives of functions and sometimes the derivatives of functions are not suitable from numerical computation point of view because of their complexity. Our objective in the present paper is to modify the N-R method by replacing the derivatives using the generalized Cauchy integral formula and selecting a suitable contour applies the standard quadrature methods. The modified N-R method has been subjected to numerical verifications.

II. MODIFICATION OF THE NEWTON-RAPHSON METHOD

Let $x \to \varphi(x)$ be a real function such that the function $\varphi(z)$,($z = x + iy$ and $i = \sqrt{-1}$) is analytic in a domain *D* which intersects the *x*-axis. Let Γ be a closed contour lying inside *D* such that Γ contains the interval *I* which is a subset of the real axis intersected by the domain *D*. Then for any point $x_0 \in I$, the generalized Cauchy integral formula is stated as

$$\varphi^{(n)}(x_n) = \frac{n!}{2\pi i}\int_{\Gamma} \frac{\varphi(z)}{(z - x_n)^{n+1}} dz \qquad (1)$$

Where the sense of description of the contour Γ is anticlockwise.

We now consider the solution of the transcendental equation $\varphi(x) = 0$ and optimization of the unimodal function $\varphi(x)$ where it is supposed that the interval of uncertainty is *I* for both the purposes. The standard N-R methods for these two purposes are prescribed respectively by the following numerical schemes:

$$x_{n+1} = x_n - \frac{\varphi(x_n)}{\varphi^{(1)}(x_n)}\ , \qquad (2)$$

$$x_{n+1} = x_n - \frac{\varphi^{(1)}(x_n)}{\varphi^{(2)}(x_n)} \qquad (3)$$

Where $\varphi^{(n)}(x)$ denotes the n^{th} derivative of $\varphi(x)$ and *n*= 0, 1, 2, . . .It is also supposed that the transcendental equation $\varphi(x) = 0$ has simple roots in the interval of uncertainty. Replacing the derivatives in (2) and (3) by using (1) we have the following schemes:

$$x_{n+1} = x_n - \frac{2\pi i \varphi(x_n)}{\int_{\Gamma} \frac{\varphi(z)}{(z - x_n)^2} dz}\ , \qquad (4)$$

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$$x_{n+1} = x_n - \frac{\int_{\Gamma} \frac{\varphi(z)}{(z-x_n)^2} dz}{2\int_{\Gamma} \frac{\varphi(z)}{(z-x_n)^3} dz} \tag{5}$$

Where x_0 is the initial guess and n=0, 1, 2, . . We now choose the contour Γ suitably in the domain of analyticity D satisfying the conditions stated above. Let Γ be a square contour with centre at x_n and vertices at the following set of points:

$$z_j = x_0 + s(1+i)i^{j+1}, j = 1(1)4 \tag{6}$$

Where s is a non-zero real number, small in magnitude, such that Γ is inside D. Let the directed line segment from the point z_j to the point z_{j+1} be denoted as L_j, where $j = 1(1)4$ and $z_5 = z_1$. Then (4) and (5) reduce to the following forms:

$$x_{n+1} = x_n - \frac{2\pi i\varphi(x_n)}{\sum_{j=1}^{4}\int_{L_j} \frac{\varphi(z)}{(z-x_n)^2} dz}, \tag{7}$$

$$x_{n+1} = x_n - \frac{\sum_{j=1}^{4}\int_{L_j} \frac{\varphi(z)}{(z-x_n)^2} dz}{2\sum_{j=1}^{4}\int_{L_j} \frac{\varphi(z)}{(z-x_n)^3} dz} \tag{8}$$

For the numerical approximation of contour integrals of analytic functions along directed line segments different methods have been formulated (cf. Milovanovic[3]) and amongst these methods the m-point transformed Gauss-Legendre rule formulated by Lether[4] is most preferable. This rule is specified as follows:

$$\int_{z_0-H}^{z_0+H} \varphi(z)dz \approx H\sum_{k=1}^{m} c_k\varphi(z_0 + H\eta_k) = Q_m(\varphi;\ z_0, H, \dots) \tag{9}$$

Where η_k are the zeros of Legendre polynomial of degree m and c_k are the associated weights in the m point G-L rule (cf. Abramoiwtz and Stegun[5]). The path of integration L_j is represented in the symmetric form as a directed line segment from the point $z_{0j} - h_j$ to $z_{0j} + h_j$ where

$$z_{0j} = (z_j + z_{j+1})/2, \quad h_j = (z_{j+1} - z_j)/2. \tag{10}$$

Now we have from (7) - (10) the following modified N-R schemes for root finding and optimization processes:

$$x_{n+1} = x_n - \frac{2\pi i\varphi(x_n)}{\sum_{j=1}^{4} Q_m(\theta_1;\ z_{0j}, h_j, x_n)}, \tag{11}$$

$$x_{n+1} = x_n - \frac{\sum_{j=1}^{4} Q_m(\theta_1;\ z_{0j}, h_j, x_n)}{\sum_{j=1}^{4} Q_m(\theta_2;\ z_{0j}, h_j, x_n)} \tag{12}$$

Respectively, where the functions θ_1 and θ_2 are given by

$$\theta_k(z) = \frac{\varphi(z)}{(z-x_n)^{k+1}}, \quad k = 1, 2. \tag{13}$$

III. SOME EXAMPLES AND NUMERICAL TESTS

The transcendental equations $\varphi(x) = 0$ which occur in Jain, Iynger and Jain [6] are solved using the scheme prescribed by (11) and (13) and the computed results have been appended in Table-I. For the purpose of computation the value of m in the m-point transformed G-L rule is considered as m=2, 3, 4, 5 and 6. It is observed that the maximum accuracy is reached in the least number of iterations by the 5- or 6-point transformed G-L rule which is same as that in case of the standard N-R technique. Computations have been carried out on MATLAB up to 15 decimal places. Similarly some functions given in Deb[2] and Joshi and Moudgalya[7] have been optimized by using the modified N-R scheme given by (12) and (13) by setting m=2 (1) 6 and the computed results have been appended in Table-II. Here too, the maximum accuracy of the

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computed results (computed up to 15 decimal places) is reached in six iterations which is same as that by the standard N-R scheme.

Table-I

$\varphi(x)$	x_0	$\varphi(x_0)$	m	min. j for max. accuracy	x_j	$\varphi(x_j)$
$2e^x - x^3 - 10x$	1.0	-5.56	2	12	0.25688382477115	0.0
			3	8	0.25688382477115	0.0
			4	6	0.25688382477115	0.0
			5	5	0.25688382477115	0.0
			6	5	0.25688382477115	0.0
	4.0	5.19	2	11	3.88863352031377	$2.1(e-14)$
			3	8	3.88863352031377	$-1.4(e-14)$
			4	6	3.88863352031377	$2.1(e-14)$
			5	6	3.88863352031377	$-1.4(e-14)$
			6	5	3.88863352031377	$-1.4(e-14)$
$cos\frac{\pi(x+1)}{8} + 0.148x$ $+0.148x - 0.9062$	-2	-0.27	2	12	-0.5081285492016	0.0
			3	10	-0.5081285492016	0.0
			4	8	-0.5081285492016	0.0
			5	8	-0.5081285492016	0.0
			6	7	-0.5081285492016	0.0
	2.0	-0.22	2	11	0.4894904492578	0.0
			3	9	0.4894904492578	0.0
			4	8	0.4894904492578	0.0
			5	7	0.4894904492578	0.0
			6	6	0.4894904492578	0.0
$cosx - x^2 - x$	1.0	-1.45	2	12	0.5500093499272	0.0
			3	9	0.5500093499272	0.0
			4	7	0.5500093499272	0.0
			5	6	0.5500093499272	0.0
			6	5	0.5500093499272	0.0
	-1.0	0.54	2	12	-1.2511518352207	0.0
			3	8	-1.2511518352207	0.0
			4	7	-1.2511518352207	0.0
			5	6	-1.2511518352207	0.0
			6	5	-1.2511518352207	0.0

(Numerical solution of transcendental equations)

Table-II

$\varphi(x)$	x_0	m	min. j for max. accuracy	x_j	$\varphi(x_j)$
$0.65 - \frac{0.75}{1+x^2}$ $-0.65tan^{-1}\frac{1}{x}$	-1.0	2	9	-1.7294858373	0.8028393584 (max)
		3	8		
		4	7		
		5	7		
		6	6		
	-1.5	2	8	-0.8647258244	1.3535756947 (min)
		3	7		
		4	6		

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$e^{x^2} + sinx + cosx + x^3$		5	6		
		6	5		
$x^2 + \frac{54}{x}$	1.0	2	8	2.999999999	27.0000000000 (min)
		3	7		
		4	6		
		5	6		
		6	6		

(Numerical optimization of unimodal functions)

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Authors’ biography

S. B. Sahoo: M. Sc. (2005), M. Phill(2006), M. Tech. in Comp. Sc. (2009), Lecturer in MCA programme in Trident Engg. College. Research interest in Computational techniques and graph theory and continuing Ph. D work in mathematics in Utkal University, Bhubaneswar. Member of Odisha Math. Soc.

Dr. M. Acharya: M. A (1992), M. Phill (1994), Ph. D. (1997), Professor. in Mathematics. in S’O’A University. Research interest: Numerical Analysis, Complex Analysis and Fluid Dynamics, Number of research Publications: 40, Guided five Ph. D. Scholars and author of four Texts in Mathematics. Member of Odisha Math. Soc. and ISI, Kolkota.

Dr. B. P. Acharya: M. Sc. (1965), Ph. D. (1972), Reted. Prof. in Mathematics, Utkal University, IIT Bhubaneswar and S’O’A Univ. Research interest: Numerical Analysis, Fluid Dynamics and OR. Number of research publications: 75 and author of 25 texts in Math. Guided 12 Ph. D scholars. Ex-President of Odisha Math. Soc.