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Dynamical Behavior in a Discrete Prey-Predator Interactions

M.ReniSagaya Raj[1], A.George Maria Selvam[2], R.Janagaraj[3].and D.Pushparajan[4]
[1,2,3] Sacred Heart College, Tirupattur - 635 601, S.India.
[4]Sri Venkateswara College of Technology, Sriperumbudur – 602 105

Abstract. This paper investigates the dynamics of a discrete-time predator - prey system. The fixed points are computed and the Jacobian associated with the fixed points are obtained. Using the eigen values, the fixed points are classified. The phase portraits show the stability of fixed points and limit cycle for a selective parameter values. Also bifurcation diagram is provided for selected range of control parameter. Numerical simulations are performed to reveal the rich complicated dynamics of the discrete model.

Key words and phrases. **Bifurcation, Difference Equations, Fixed Points, Predator - Prey System, Stability.**

I. INTRODUCTION

Population dynamics has been a dominant branch of theoretical ecology. Population modeling has several important applications in species management: managing fisheries to determine sustainable yields, formulating policies to save species threatened by extinction (Project Tiger-India, sparrows, endangered vultures), or trying to prevent the spread of invasive species (Africanized Honeybee, Amynthas). The Lotka – Volterra prey – predator model is one of the fundamental population models describing the species interactions in a closed habitat. In 1926, Vito Volterra formulated a model to describe the fluctuations that had been observed in the predator (shark) and prey fish population in the Adriatic sea. In 1926, A.J.Lotka (an American Biophysicist) constructed a similar model in a different context. In Lotka – Volterra model no population can dominate and there is no possibility of either population becoming extinct. The system exhibits periodic oscillations [2], [5], [6]. Also the system is structurally unstable. The model makes several assumptions: The prey is the main unlimited source of food for the predator population. In the absence of the predator, the prey population grows exponentially. In the absence of prey, the predator population declines and goes extinct. Several authors improved the basic Lotka – Volterra model by introducing Allee effect and functional responses [10], [11], [12].

II. MODEL OF INTERACTIONS AND FIXED POINTS

The discrete time models governed by difference equations are more appropriate when the populations have non overlapping generations. Discrete models can also provide efficient computational models of continuous models for numerical simulations. The maps defined by simple difference equation can lead to rich complicated dynamics [1], [4], [8], [14], [15]. This paper considers the following system of difference equations which describes interactions between two species.

$$\begin{aligned} x(n+1) &= rx(n)[1-x(n)] - ax(n)y(n) \\ y(n+1) &= (1-c)y(n) + bx(n)y(n),\; x(0)>0,\; y(0)>0. \end{aligned} \tag{1}$$

Where *r, a, b, c > 0*. The prey population follows logistic growth. The fixed points of the system (1) are calculated by solving the following algebraic system.

$x = rx[1-x] - axy,\;\; y = (1-c)y + bxy$

A simple straightforward computation gives the following fixed points.

(a) $E_0 = (0, 0)$, origin (both species go extinct).

(b) $E_1 = \left(\frac{r-1}{r}, 0\right)$ is the axial fixed point (extinction of predator).

(c) Interior fixed point $E_2 = \left(\frac{c}{b}, \frac{r}{b}\left(1-\frac{c}{b}\right) - \frac{1}{a}\right)$ (coexistence of both species). The interior point E_2 is a positive equilibrium point provided $r > \frac{b}{b-c}$.

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III. CLASSIFICATION AND STABILITY OF FIXED POINTS

Nonlinear systems are much harder to analyze since in most cases they do not possess quantitative solutions. Even when explicit solutions are available, they are often too complicated to provide much insight. Hence mathematicians prefer to analyze nonlinear systems qualitatively. One of the most useful techniques for analyzing nonlinear systems qualitatively is the linearised stability technique. The stability of the system is investigated by obtaining the eigen values of the Jacobian matrix associated with fixed points [7], [9]. A fixed point is stable if all sufficiently small perturbations away from it damp out in time. The following lemma [13] is useful in obtaining nature of fixed points by conditions imposed by parameters.

Lemma 1. Let $p(\lambda)=\lambda^2-B\lambda+C$ and λ_1,λ_2 be the roots of $p(\lambda)=0$. Suppose that *p(1) > 0*. Then we have

i. $|\lambda_1|<1$ and $|\lambda_2|<1$ if and only if $p(-1)>0$ and $C<1$.

ii. $|\lambda_1|<1$ and $|\lambda_2|>1$ (or $|\lambda_1|>1$ and $|\lambda_2|<1$) if and only if $p(-1)<0$.

iii. $|\lambda_1|>1$ and $|\lambda_2|>1$ if and only if $p(-1)>0$ and $C>1$.

iv. $|\lambda_1|=-1$ and $|\lambda_2|\neq 1$ if and only if $p(-1)=0$ and $B\neq 0,2$.

v. λ_1 and λ_2 are complex and $|\lambda_1|=|\lambda_2|$ if and only if $B^2-4C<0$ and $C=1$.

The characteristic roots λ_1 and λ_2 of $p(\lambda)=0$ are called eigen values of the fixed point (x^*,y^*). Then the fixed point (x^*,y^*) is a sink if $|\lambda_{1,2}|<1$. Hence the sink is locally asymptotically stable. The fixed point (x^*,y^*) is a source if $|\lambda_{1,2}|>1$. The source is locally unstable. The fixed point (x^*,y^*) is a saddle if $|\lambda_1|<1$ and $|\lambda_2|>1$ (or $|\lambda_1|>1$ and $|\lambda_2|<1$). Finally (x^*,y^*) is called non hyperbolic if either $|\lambda_1|=1 or |\lambda_2|=1$. For the system (1), we have the following results [6].

The Jacobian matrix J for the system (1) is

$$J(x,y)=\begin{pmatrix} r-2rx-ay & -ax \\ by & bx-c+1 \end{pmatrix}$$

The characteristic polynomial is $p(\lambda)=\lambda^2-\lambda Tr\,J+Det\,J$. Even though both species becoming extinct do not attract much, we will give conditions under which the populations become extinct. The Jacobian at E_0 is of the form

$$J(E_0)=\begin{pmatrix} r & 0 \\ 0 & 1-c \end{pmatrix}$$

At E_0, the eigen values are $\lambda_1=r$ and $\lambda_2=1-c$.

Proposition 1. The fixed point E_0 is a

- sink if $r<1$ and $0<c<2$,
- saddle if $r<1$ and $c>2$,
- source if $r>1$ and $c>2$,
- non hyperbolic if either $r=1$ or $c=2$.

The Jacobian matrix for E_1 is

$$J(E_1)=\begin{pmatrix} 2-r & a\left(\frac{1-r}{r}\right) \\ 0 & b\left(\frac{r-1}{r}\right)-c+1 \end{pmatrix}$$

At E_1, the eigen values are $\lambda_1=2-r$ and $\lambda_2=b\left(\frac{r-1}{r}\right)-c+1$

Proposition 2. The fixed point E_1 is a

- sink if $1<r<3$ and $b<\frac{cr}{r-1}$, source if $r>3$ and $b>\frac{cr}{r-1}$, saddle if $1<r<3$ and $b>\frac{cr}{r-1}$.

The Jacobian for the interior equilibrium point E_2 is

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$$J(E_2)=\begin{pmatrix} 1-\dfrac{rc}{b} & -\dfrac{ac}{b} \\ \dfrac{r(b-c)-b}{a} & 1 \end{pmatrix}$$

Computation yields $Tr=2-\dfrac{rc}{b}$ and $Det=cr\left(1-\dfrac{(1+c)}{b}\right)-c+1$. At E_2, the characteristic polynomial is $p(\lambda)=\lambda^2-\left(2-\dfrac{rc}{b}\right)\lambda+cr\left(1-\dfrac{(1+c)}{b}\right)-c+1$. Also $\lambda_{1,2}=1-\dfrac{rc}{2b}\pm\sqrt{\left(1-\dfrac{rc}{2b}\right)^2-c\left[r+1-\dfrac{r}{b}(1+c)\right]-1}$

Proposition 3. The fixed point E_2 is a

- sink if $\dfrac{b(c-4)}{c(b-(2+c))}<r<\dfrac{b}{b-(1+c)}$, source if $r>\dfrac{b(c-4)}{c(b-(2+c))}$ and $r>\dfrac{b}{b-(1+c)}$,
- saddle if $r<\dfrac{b(c-4)}{c(b-(2+c))}$.

IV. NUMERICAL SIMULATIONS

In this section, numerical simulations are presented to illustrate some results of the previous sections which exhibit rich dynamical nature of the model. Mainly, we present the orbits of the solutions x and y with phase plane diagrams (sinks, limit cycle and bifurcation) for the predator-prey system (1). Dynamic nature of the system (1) about the equilibrium points under different sets of parameter values are presented in this section. Existence of limit cycle for selective set of parameters is established through phase plane in Figure-4. Also the bifurcation diagram (Figure-5) indicates the existence of chaos in both prey and predator populations [3].

Example 1. *For the values r = 0.99, a = 0.01, b = 0.095, c = 0.04, the Eigen values are* λ_1 *= 0.99 and* λ_2 *= 0.96 so that* $|\lambda_{1,2}|<1$. *Hence the trivial fixed point* E_0 *is stable. The orbit and the phase diagram illustrate the result, see Figure - 1.*

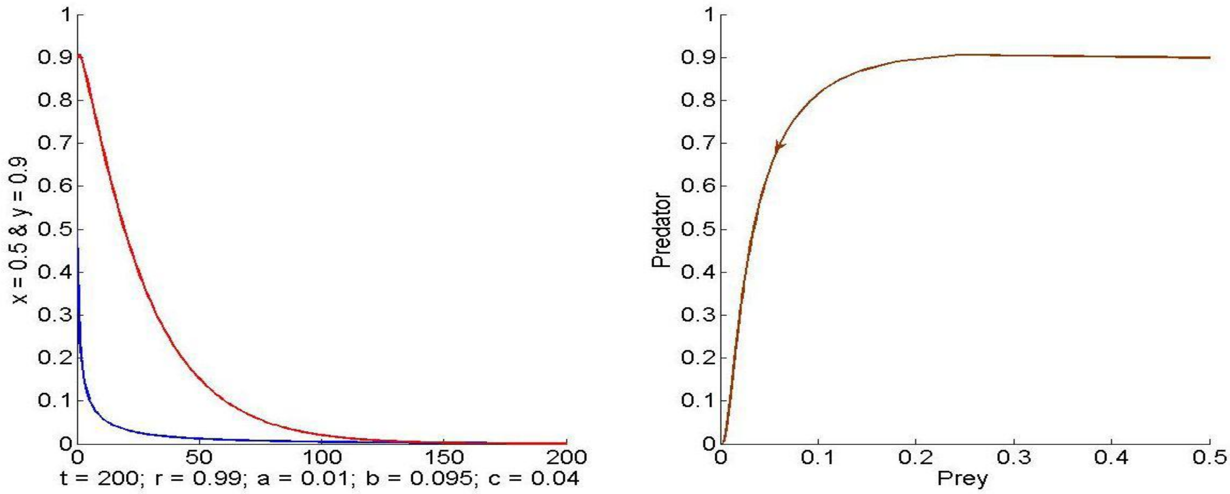

Fig 1. Stability at E_0

Example 2. *For the values r = 2.98, a = 0.09, b = 1.59, c = 1.1, we obtain* E_1 *= (0.66, 0) which is an axial fixed point. Eigen values are* $\lambda_1=-0.98$ *and* λ_2 *= 0.96 so that* $|\lambda_{1,2}|<1$. *Hence the fixed point is stable. The time plot and the phase diagram illustrate the result, see Figure - 2.*

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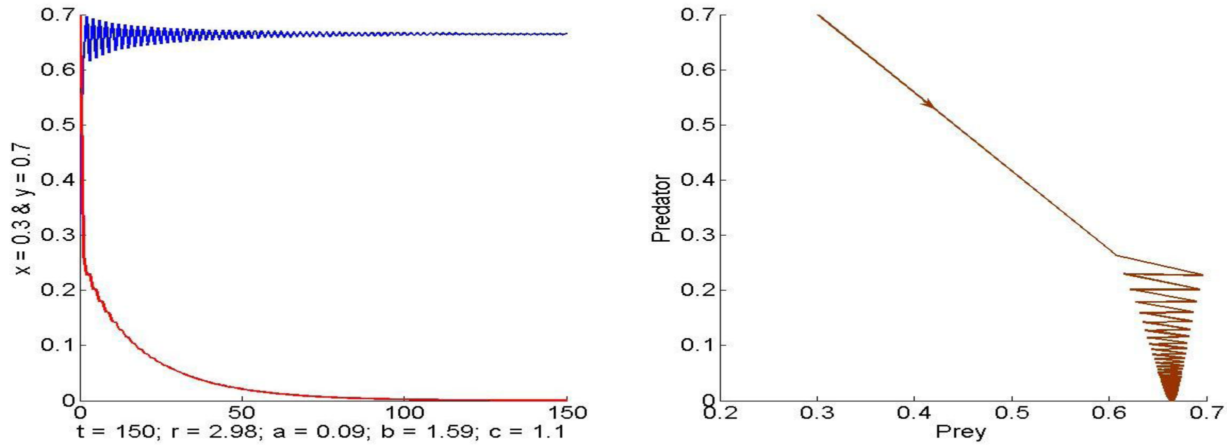

Fig 2. Stability at E_1

Example 3. *In this example, we take $r = 2.5$, $a = 1.5$, $b = 3.3$ and $c = 0.99$. Computations yield $E_2 = (0.3, 0.5)$. The eigen values are $\lambda_{1,2} = 0.6250 \pm i\ 0.7758$ and $|\lambda_{1,2}| = 0.9962 < 1$. Hence the criteria for stability are satisfied. The phase portrait in Figure - 3 shows a sink and the trajectory spirals towards the fixed point E_2.*

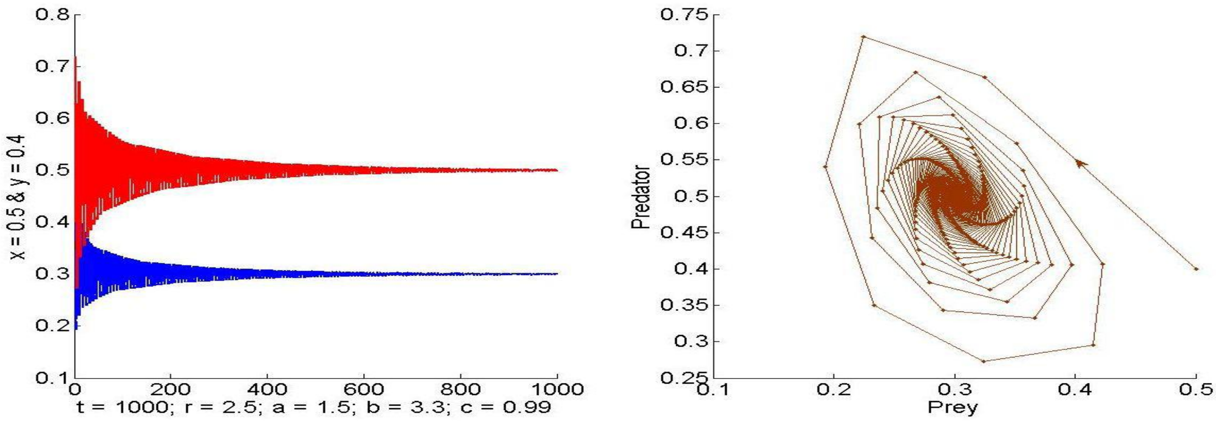

Fig 3. Stability at E_2

Example 4. *The parameters are $r = 2.55$, $a = 1.3$, $b = 3.35$, $c = 0.99$. The initial conditions on the populations of the species are $x(0) = 0.5$ and $y(0) = 0.4$. The trajectory spirals inwards but does not approach a point. The trajectory finally settles down as a limit cycle, see Figure-4.*

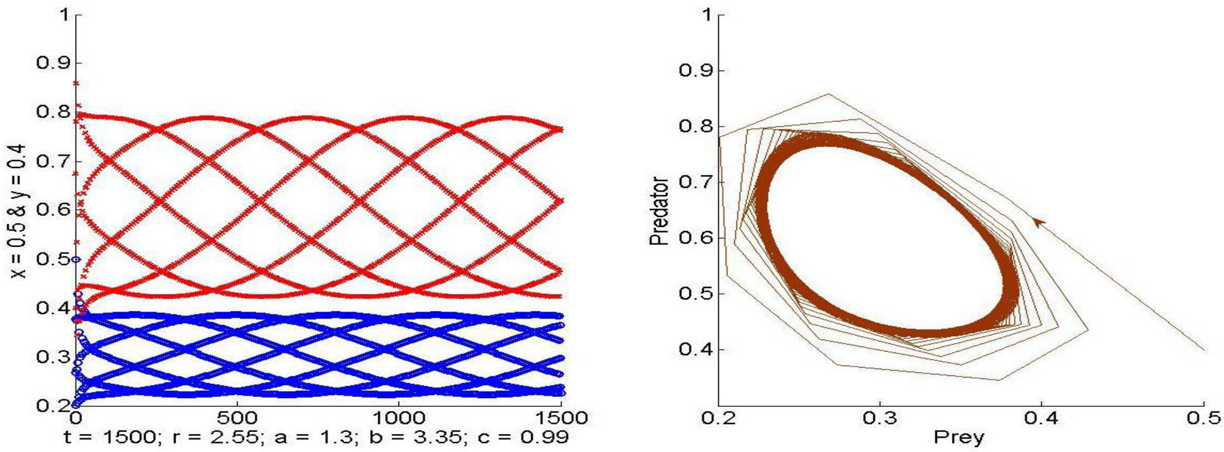

Fig 4. Limit Cycle

The qualitative behavior of the system can change as parameters are varied. In particular, stability of the fixed points can change. These qualitative changes in the dynamics are called bifurcation. In short qualitative changes are tied with bifurcation. The parametric values at which they occur are called bifurcation points. In this study the control parameter is *r*.

Example 5. *The parameters are assigned the values a = 1.3, b = 3.35, c = 0.99 and the bifurcation diagram is plotted for the growth parameter r in the range 2 – 3.8. Both prey and predator population undergoes chaos, see Figure-5.*

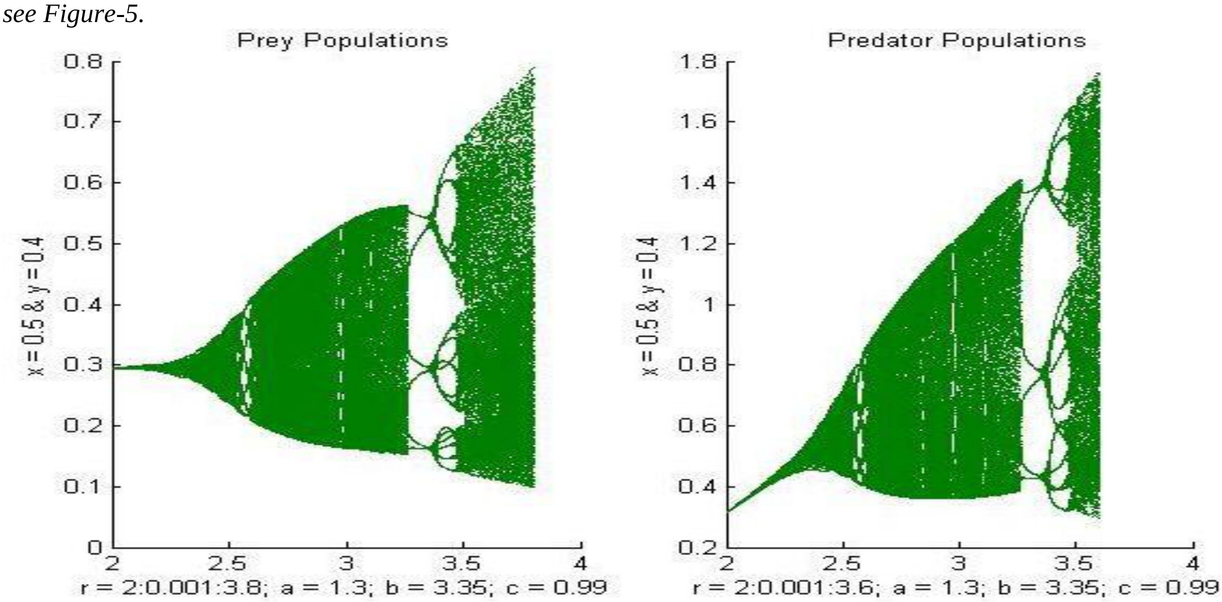

Fig 5. Bifurcations Diagram

V. CONCLUSION

This paper investigated the complex dynamical behavior of a 2-dimensional discrete predator - prey model. Fixed points are computed and stability conditions are obtained. The fixed points are classified and the conditions are obtained in terms of the parameters (Propositions 1, 2, 3). The results are illustrated with suitable hypothetical sets of parameter values. Numerical simulations are presented to show the dynamical behavior of the system (1). Finally, bifurcation diagrams for both species presented shows the existence of chaos.

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