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Modified H^1-Galerkin Mixed Finite Element Method for Damped Sine-Gordon Equation

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***Abstract*—The damped Sine-Gordon equation had been studied by H^1-Galerkin mixed finite element method, but it is a pity that the $L^\infty(L^2)$-norm estimate of auxiliary variable q in two and three space dimensions is suboptimal. We propose the modified H^1-Galerkin semi-discrete and fully-discrete schemes for the model equation in two and three space dimensions, and optimal error estimates are derived for both schemes without the LBB-consistency condition and the restriction on the approximating spaces.**

***Index Terms*—Damped Sine-Gordon Equation, Modified H^1-Galerkin Mixed Finite Element Method, Fully-Discrete Scheme, Optimal Error Estimate.**

I. INTRODUCTION

In this paper, we consider the following damped Sine-Gordon equation with Dirichlet boundary condition and initial conditions

$$\begin{cases} u_{tt} + \kappa u_t - \Delta u + \sin u = 0, & (x,t) \in \Omega \times J, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \bar{J}, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

Where Ω is a bounded convex polygonal domain in $R^d(d = 2,3)$ with Lipschitz continuous boundary $\partial\Omega$, $J = (0,T]$ with $0 < T < \infty$, and the parameter κ is the so-called dissipative term which is assumed to be real number with $\kappa \ge 0$. Assume that the initial functions $u_0(x), u_1(x)$ are sufficiently smooth. In order to get the modified H^1-Galerkin mixed finite element procedure, we rewrite the equation into first order system by introducing new variable with the physical quantity of interest. With $q = \nabla u$, we reformulate the equation (1) as the first order system

$$\begin{cases} \nabla u = q, \\ u_{tt} + \kappa u_t - \nabla \cdot q + \sin u = 0. \end{cases} \tag{2}$$

In 1998, Pani (see Ref. [1]) proposed a new mixed finite element method called H^1-Galerkin mixed finite element procedure for parabolic partial differential equations. It not only takes advantage of the least-square method and classical mixed finite element method but also yields a better rate of convergence. And more important, the H^1-Galerkin mixed finite element is not subject to LBB-consistency condition and the finite element spaces V_h (for approximating the original variable u) and W_h (for approximating the auxiliary variable q) are allowed to be of different polynomial degrees. The H^1-Galerkin method was applied to parabolic integro-differential equation (Ref. [2]), hyperbolic equation (Ref. [3]), pseudo-hyperbolic (Ref. [4]) and so on. The damped Sine-Gordon equation (1) was studied in Ref. [5], and optimal error estimates are derived for both semi-discrete and fully-discrete schemes in one space dimension, but it is a pity that the $L^\infty(L^2)$-norm estimate of auxiliary variable q in two and three space dimensions is suboptimal. In order to obtain the optimal estimates for q in $L^\infty(L^2)$-norm without restricting the finite element spaces W_h, we propose a modified H^1-Galerkin mixed finite element schemes for problems (1). And the optimal error estimates for both semi-discrete and implicit fully-discrete schemes in two and three space variables are proved. The brief outline of this paper is as follows. In section 2, a modified semi-discrete H^1-Galerkin mixed finite element procedure is constructed without restricting the approximating spaces for q in two and three space dimensions and optimal estimates are derived. An implicit fully-discrete formulation is

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described in section 3, and also we obtain the optimal error estimates. Throughout this paper, C will denote a generic positive constant which does not depend on the spatial mesh parameter h and time discretization parameter Δt.

II. MODIFIED SEMI-DISCRETE H^1-GALERKIN MIXED FINITE ELEMENT PROCEDURE

Denote the natural inner product in $L^2(\Omega)$ by $(\cdot,\cdot)$. Let $H_0^1=\{v\in H^1(\Omega): v=0 \text{ on } \partial\Omega\}$. $\mathbf{H}^m=(H^m)^d$ (d=2 or 3) denotes the corresponding product space with usual product norm. $W^{m,p}(\Omega)(1\le p\le\infty)$ denotes the classical Sobolev spaces with norm $\|\cdot\|_{m,p}$. And when p=2, we simply write $W^{m,p}(\Omega)$ as H^m with norm $\|\cdot\|_m$. Now, with $q=\nabla u$, use the fact that $\nabla\times\nabla u=0$, and hence we get

$$\nabla\times q=0. \tag{3}$$

Then, having in mind the boundary condition in equation (1), we get (see Ref. [6])

$$n\wedge\nabla u=n\wedge q=0 \text{ on } \partial\Omega, \tag{4}$$

Where n is the outward normal and $\wedge$ denotes the exterior product. Add (3) and (4) to the first order system (2). More precisely, with $q=\nabla u$, we rewrite (2) as

$$\begin{cases}\nabla u=q,(x,t)\in\Omega\times J,\\ u_{tt}+\kappa u_t-\nabla\cdot q+\sin u=0,(x,t)\in\Omega\times J,\\ \nabla\times q=0,(x,t)\in\Omega\times J,\\ n\wedge q=0,(x,t)\in\partial\Omega\times J,\\ u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega.\end{cases} \tag{5}$$

For the weak formulation, let $W=\{w\in(H^1)^d: n\wedge w=0 \text{ on } \partial\Omega\}$, d=2 or 3. Using Gauss divergence theorem, we now seek a pair $\{u,q\}:[0,T]\mapsto H_0^1\times W$ such that

$$(\nabla u,\nabla v)=(q,\nabla v), v\in H_0^1, \tag{6a}$$

$$(q_{tt},w)+\kappa(q_t,w)+A(q,w)=-(q\cos u,w), w\in \mathbf{H}^1, \tag{6b}$$

where $A(\phi,w)=(\nabla\cdot\phi,\nabla\cdot w)+(\nabla\times\phi,\nabla\times w)$.

To define the modified semi-discrete H^1-Galerkin mixed finite element procedure, let T_h be a partition of Ω into a finite number of elements called simplexes, i.e., $\Omega=\cup_{K\in T_h}K$ with $h=\{diam(K): K\in T_h\}$. Let V_h be finite dimensional subspace of H_0^1 satisfying the following approximation property: for k positive integer

$$\inf_{v_h\in V_h}\{\|v-v_h\|+h\|v-v_h\|_1\}\le Ch^{k+1}\|v\|_{k+1}, v\in H^{k+1}\cap H_0^1.$$

Standard example of such space is as follow

$$V_h=\{v_h\in C^0(\Omega): v_h|_k\in P_k(K),\forall K\in T_h, v_h=0 \text{ on } \partial\Omega\}.$$

Then define the finite element space W_h of the space W by

$$W_h=\{w_h\in(C(\overline{\Omega}))^d:(w_h)_i|_K\in P_r(K), i=1,2,\cdots,d,\forall K\in T_h, n\wedge w_h=0, \text{at the nodes on} \partial\Omega\}.$$

Since $n\wedge w_h=0$ only at the boundary nodes, the finite element space W_h is not a subspace of W and hence, it results in a non-conforming method. Note that the finite dimensional space W_h satisfies the approximation property (see Ref. [6], [7]): for positive integer r

$$\inf_{w_h\in W_h}\{\|w-w_h\|+h\|w-w_h\|_1\}\le Ch^{r+1}\|w\|_{r+1}, w\in \mathbf{H}^{r+1}\cap \mathbf{H}^1.$$

The modified semi-discrete H^1-Galerkin mixed finite element approximation for (6) is determined as a pair

$\{u_h, q_h\}:[0,T] \mapsto V_h \times W_h$ satisfying

$$(\nabla u_h, \nabla v_h) = (q_h, \nabla v_h), v_h \in V_h, \tag{7a}$$

$$(q_{htt}, w_h) + \kappa(q_{ht}, w_h) + A(q_h, w_h) = -(q_h \cos u_h, w_h), w_h \in W_h. \tag{7b}$$

With appropriately chosen initial pair $\{q_h(0), q_{ht}(0)\}$ to be defined later. For the error analysis, we define the auxiliary projections $\{\tilde{u}_h, \tilde{q}_h\}:[0,T] \mapsto V_h \times W_h$ by

$$(\nabla(u - \tilde{u}_h), \nabla v_h) = 0, v_h \in V_h, \tag{8a}$$

$$A_1(q - \tilde{q}_h, w_h) = 0, w_h \in W_h, \tag{8b}$$

Where $A_1(\phi, w_h) = A(\phi, w_h) + (\phi, w_h)$ .When the domain Ω is convex or the boundary $\partial\Omega$ is of class $C^{1,1}$ or Ω is a curvilinear polygon (or polytope) of class $C^{1,1}$ with no concave angles, then there is a positive constant μ_0 independent of h such that following estimate holds

$$\| q_h \|^2_{H(div,\Omega)} + \| \nabla \times q_h \|^2 \geq \mu_0 \| q_h \|^2_1,$$

for all $q_h \in W_h$ and for small h (see Ref. [6]). Thus, $A_1(\cdot,\cdot)$ satisfies the coercivity condition

$$A_1(\phi_h, \phi_h) \geq \mu_0 \| \phi_h \|^2_1, \phi_h \in W_h. \tag{9}$$

Let $u - \tilde{u}_h = \eta$ and $q - \tilde{q}_h = \rho$. With an appropriate modification of the analysis of Pehlivanov and Carey (see Ref. [6]), the following estimates for ρ and its temporal derivatives are easy to derive.

$$\sum_{i=0}^{j} \| \frac{\partial^i \rho}{\partial t^i} \| \leq Ch^{r+1} \sum_{i=0}^{j} \| \frac{\partial^i q}{\partial t^i} \|_{r+1}, j = 0,1,2. \tag{10}$$

And we obtain (see Ref. [8])

$$\| \eta \| + h \| \eta \|_1 \leq Ch^{k+1} \| u \|_{k+1}. \tag{11}$$

For semi-discrete error estimates, we now split the errors $u - u_h = (u - \tilde{u}_h) + (\tilde{u}_h - u_h) = \eta + \zeta$ and $q - q_h = (q - \tilde{q}_h) + (\tilde{q}_h - q_h) = \rho + \xi$. Below, we prove our main theorem in this section.

Theorem 2.1. Assume that $q_h(0) = \tilde{q}_h(0)$ with $q(0) = \nabla u_0$ so that $\xi(0) = 0$. Further, let $q_{ht}(0) = P_h q_t(0)$, where P_h is the L^2 projection defined by $(w - P_h w, w_h) = 0, \forall w_h \in W_h$. Then there exists a positive constant C independent of h such that

$$\|(q - q_h)(t)\| \leq Ch^{\min\{r+1,k+1\}}(\|*\| + \| q \|_{L^\infty(H^{r+1})}),$$

$$\|(q - q_h)(t)\|_1 \leq Ch^{\min\{r,k+1\}}(\|*\| + \| q \|_{L^\infty(H^{r+1})}),$$

$$\|(u - u_h)(t)\| \leq Ch^{\min\{r+1,k+1\}}(\|*\| + \| q \|_{L^\infty(H^{r+1})} + \| u \|_{L^\infty(H^{k+1})}),$$

$$\|(u - u_h)(t)\|_1 \leq Ch^{\min\{r+1,k\}}(\|*\| + \| q \|_{L^\infty(H^{r+1})} + \| u \|_{L^\infty(H^{k+1})}),$$

Where $\|*\| = (1 + \| q \|_{L^\infty(L^\infty)})[\| q_t(0) \|_{r+1} + \sum_{i=0}^{2} \| \frac{\partial^i q}{\partial t^i} \|_{L^2(H^{r+1})} + \| q \|_{L^\infty(L^\infty)}(\| q \|_{L^2(H^{r+1})} + \| u \|_{L^2(H^{k+1})})]$.

Proof. From (6)-(7) with the projections (8), we have

$$(\nabla \zeta, \nabla v_h) = (\rho + \xi, \nabla v_h). \tag{12a}$$

$$\begin{aligned}(\xi_{tt}, w_h) + \kappa(\xi_t, w_h) + A_1(\xi, w_h) &= -(\rho_{tt}, w_h) - \kappa(\rho_t, w_h) \\ &+ (\rho + \xi, w_h) - (q \cos u - q_h \cos u_h, w_h).\end{aligned} \tag{12b}$$

Choose $v_h = \zeta$ in (12a) and use $\| \zeta \|_1 \leq C \| \nabla \zeta \|$ as $\zeta \in H^1_0$ to obtain

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$$\|\zeta\| \le \|\zeta\|_1 \le C\|\nabla\zeta\| \le C(\|\rho\|+\|\xi\|). \tag{13}$$

Further, setting $w_h = \xi_t$ in (12b), we get

$$\begin{aligned}(\xi_{tt},\xi_t)+\kappa(\xi_t,\xi_t)+A_1(\xi,\xi_t) &= -(\rho_{tt},\xi_t)-\kappa(\rho_t,\xi_t)\\ &+(\rho+\xi,\xi_t)-(q(\cos u-\cos u_h)+(q-q_h)\cos u_h,\xi_t).\end{aligned} \tag{14}$$

Noting that $\cos u-\cos u_h = -2\sin((u+u_h)/2)\sin((u-u_h)/2)$, apply the Cauchy-Schwarz's inequality and the Young's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}(\|\xi_t\|^2+A_1(\xi,\xi))+\kappa\|\xi_t\|^2 \le C\|q\|^2_{L^\infty(L^\infty)}(\|\eta\|^2+\|\zeta\|^2)+C(\|\rho_{tt}\|^2+\|\rho_t\|^2+\|\rho\|^2+\|\xi\|^2)+\kappa\|\xi_t\|^2. \tag{15}$$

Substituting (13) into (15), integrating with respect to time, noting that the positive definiteness of A_1 and $\xi(0)=0$, we can easily get

$$\begin{aligned}\|\xi_t\|^2+\|\xi\|_1^2 &\le C\|\xi_t(0)\|^2+C\int_0^t(\|\rho_{tt}\|^2+\|\rho_t\|^2+\|\rho\|^2)ds\\ &+C\|q\|^2_{L^\infty(L^\infty)}\int_0^t(\|\eta\|^2+\|\rho\|^2)ds+C(1+\|q\|^2_{L^\infty(L^\infty)})\int_0^t\|\xi\|^2ds.\end{aligned} \tag{16}$$

Using Gronwall's lemma and noting the fact that $\int_0^t\int_0^\tau|\psi(s)|^2\,dsd\tau \le C\int_0^t|\psi(s)|^2\,ds$ where ψ is a integrable function in [0,T], we get

$$\|\xi\|_1^2 \le \|q\|^2_{L^\infty(L^\infty)}\int_0^t(\|\rho\|^2+\|\eta\|^2)ds]+C(1+\|q\|^2_{L^\infty(L^\infty)})[\|\xi_t(0)\|^2+\int_0^t(\|\rho_{tt}\|^2+\|\rho_t\|^2+\|\rho\|^2)ds. \tag{17}$$

Further, it follows from L^2-projection of $q_t(0)$ that

$$\|\xi_t(0)\| \le \|q_t(0)-q_{ht}(0)\|+\|q_t(0)-\tilde{q}_{ht}(0)\| \le Ch^{r+1}\|q_t(0)\|_{r+1}. \tag{18}$$

Using (10), (17) and (18), we obtain the super convergence result of ξ in H^1-norm

$$\|\xi\|_1 \le Ch^{\min\{r+1,k+1\}}(1+\|q\|_{L^\infty(L^\infty)})[\|q_t(0)\|_{r+1}+\sum_{i=0}^{2}\|\frac{\partial^i q}{\partial t^i}\|_{L^2(H^{r+1})}+\|q\|_{L^\infty(L^\infty)}(\|q\|_{L^2(H^{r+1})}+\|u\|_{L^2(H^{k+1})})]. \tag{19}$$

Apply the above superconvergence result in (13) with estimates (10) to obtain

$$\|\zeta\| \le \|\zeta\|_1 \le Ch^{\min\{r+1,k+1\}}[(1+\|q\|_{L^\infty(L^\infty)})(\|q_t(0)\|_{r+1}+\sum_{i=0}^{2}\|\frac{\partial^i q}{\partial t^i}\|_{L^2(H^{r+1})}+\|q\|_{L^\infty(L^\infty)}(\|q\|_{L^2(H^{r+1})}+\|u\|_{L^2(H^{k+1})}))+\|q\|_{L^\infty(H^{r+1})}]. \tag{20}$$

Finally, apply the triangle inequality with (10), (11), (19) and (20) to complete the proof.

Remark 2.1. (i) When $k \le r$, we obtain the optimal order of convergence for $u-u_h$ in $L^\infty(L^2)$ and $L^\infty(H^1)$ norm with respect to the approximation property of the finite element spaces.

(ii) When $k \ge r$, we have $\|q-q_h\|_{L^\infty(L^2)} = O(h^{r+1})$ and $\|q-q_h\|_{L^\infty(H^1)} = O(h^r)$. Compared to Ref. [5], the present analysis indicates that we obtain an optimal estimate of $q-q_h$ in $L^\infty(L^2)$ and $L^\infty(H^1)$ norm with respect to the approximation property of the finite element space W_h.

(iii)Assume that d=2 (that is $\Omega\subset R^2$), $q_h(0)=\tilde{q}_h(0)$, and the finite element mesh is quasi-uniform, using Sobolev imbedding theorem and superconvergence estimate (20), we get

$$\begin{aligned}\|\zeta\|_{L^\infty} &\le Ch^{\min\{k+1,r+1\}}|\ln h|^{\frac{1}{2}}[(1+\|q\|_{L^\infty(L^\infty)})(\|q_t(0)\|_{r+1}+\sum_{i=0}^{2}\|\frac{\partial^i q}{\partial t^i}\|_{L^2(H^{r+1})}\\ &+\|q\|_{L^\infty(L^\infty)}(\|q\|_{L^2(H^{r+1})}+\|u\|_{L^2(H^{k+1})}))+\|q\|_{L^\infty(H^{r+1})}].\end{aligned}$$

From Ref. [9-10], using the quasi-uniformity condition of the finite element mesh, we obtain the estimate for η in the elliptic projection

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$$\|\eta\|_{L^\infty} \le \begin{cases} C|\ln h|h^2\|u\|_{L^\infty(W^{K+1,\infty})}, & k=1, \\ Ch^{k+1}\|u\|_{L^\infty(W^{K+1,\infty})}, & k>1. \end{cases}$$

Apply the triangle inequality to obtain an quasi-optimal maximum norm estimates

$$\|u-u_h\|_{L^\infty} \le C|\ln h|h^{\min\{k+1,r+1\}}[(1+\|q\|_{L^\infty(L^\infty)})(\|q_t(0)\|_{r+1} + \sum_{i=0}^{2}\|\frac{\partial^i q}{\partial t^i}\|_{L^2(H^{r+1})}$$
$$+\|q\|_{L^\infty(L^\infty)}(\|q\|_{L^2(H^{r+1})} + \|u\|_{L^2(H^{k+1})})) + \|q\|_{L^\infty(H^{r+1})} + \|u\|_{L^\infty(W^{K+1,\infty})}].$$

III. MODIFIED FULLY-DISCRETE H^1-GALERKIN MIXED FINITE ELEMENT PROCEDURE

In this section, we briefly describe a fully discrete scheme for approximating a pair of solution $\{u,q\}$ of the system (2) and discuss a priori error bounds.

Let $0=t_0<t_1<\cdots<t_N=T$ be a given partition of the time interval [0,T] with step length $\Delta t = T/N$, for some positive integer N and define $t_n = n\Delta t$. We use the following notation related to functions defined at discrete time levels. For a given function ϕ on [0,T], let

$$\phi^n = \phi(t_n), \phi^{n+\frac{1}{2}} = \frac{1}{2}(\phi^{n+1}+\phi^n), \partial_t\phi^{n+\frac{1}{2}} = \frac{\phi^{n+1}-\phi^n}{\Delta t}, \bar{\partial}_t\phi^n = \frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t},$$

$$\partial_t^2\phi^n = \frac{\partial_t\phi^{n+\frac{1}{2}} - \partial_t\phi^{n-\frac{1}{2}}}{\Delta t}, \phi^{n;\frac{1}{4}} = \frac{1}{4}(\phi^{n+1}+2\phi^n+\phi^{n-1}) = \frac{1}{2}(\phi^{n+\frac{1}{2}}+\phi^{n-\frac{1}{2}}).$$

Let U^n and Z^n respectively, be the approximations of u and q at $t=t_n$ which we define through the following implicit scheme. We now determine a sequence of pairs $\{U^n, Z^n\} \in V_h \times W_h, n=0,1,\cdots,N$, satisfying

$$\begin{aligned} &(U^0, v_h) = (u_0, v_h), (Z^0, w_h) = (q(0), w_h), v_h \in V_h, w_h \in W_h, \\ &(\nabla U^{n+\frac{1}{2}}, \nabla v_h) = (Z^{n+\frac{1}{2}}, \nabla v_h), v_h \in V_h, n \ge 0, \\ &(\frac{2}{\Delta t}\partial_t Z^{\frac{1}{2}}, w_h) + \kappa(\partial_t Z^{\frac{1}{2}}, w_h) + A(Z^{\frac{1}{2}}, w_h) = -((Z\cos U)^{\frac{1}{2}}, w_h) + (\frac{2}{\Delta t}q_t(0), w_h), \\ &(\partial_t^2 Z^n, w_h) + \kappa(\bar{\partial}_t Z^n, w_h) + A(Z^{n;\frac{1}{4}}, w_h) = -((Z\cos U)^{n;\frac{1}{4}}, w_h), w_h \in W_h, n \ge 1, \end{aligned} \tag{21}$$

here, $q(0) = \nabla u_0$ and $q_t(0) = \nabla u_1$. Now we split the error $u(t_n) - U^n = (u(t_n) - \tilde{u}_h(t_n)) + (\tilde{u}_h(t_n) - U^n) = \eta^n + \zeta^n$ and $q(t_n) - Z^n = (q(t_n) - \tilde{q}_h(t_n)) + (\tilde{q}_h(t_n) - Z^n) = \rho^n + \xi^n$, where $\tilde{u}_h$ and $\tilde{q}_h$ are defined by (8). Using (8), we obtain the equations for ζ^n and ξ^n as following: for $v_h \in V_h$ and $w_h \in W_h$

$$(\nabla\zeta^{n+\frac{1}{2}}, \nabla v_h) = (\rho^{n+\frac{1}{2}} + \xi^{n+\frac{1}{2}}, \nabla v_h), n \ge 0, \tag{22a}$$

$$\begin{aligned} (\frac{2}{\Delta t}\partial_t\xi^{\frac{1}{2}}, w_h) + \kappa(\partial_t\xi^{\frac{1}{2}}, w_h) + A_1(\xi^{\frac{1}{2}}, w_h) &= -(\tau^0, w_h) - (\frac{2}{\Delta t}\partial_t\rho^{\frac{1}{2}}, w_h) \\ &-\kappa(\beta^0, w_h) - \kappa(\partial_t\rho^{\frac{1}{2}}, w_h) + (\rho^{\frac{1}{2}} + \xi^{\frac{1}{2}}, w_h) - ((q\cos u)^{\frac{1}{2}} - (Z\cos U)^{\frac{1}{2}}, w_h), \end{aligned} \tag{22b}$$

$$\begin{aligned} (\partial_t^2\xi^n, w_h) + \kappa(\bar{\partial}_t\xi^n, w_h) + A_1(\xi^{n;\frac{1}{4}}, w_h) &= -(\tau^n, w_h) - (\partial_t^2\rho^n, w_h) - \kappa(\beta^n, w_h) \\ &-\kappa(\bar{\partial}_t\rho^n, w_h) + (\rho^{n;\frac{1}{4}} + \xi^{n;\frac{1}{4}}, w_h) - ((q\cos u)^{n;\frac{1}{4}} - (Z\cos U)^{n;\frac{1}{4}}, w_h), \end{aligned} \tag{22c}$$

Where

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$$\tau^0 = q_{tt}^{\frac12} + \frac{2}{\Delta t}(q_t(0) - \partial_t q^{\frac12}),\ \tau^n = q_{tt}^{n;\frac14} - \partial_t^2 q(t_n), \beta^0 = q_t^{\frac12} - \partial_t q^{\frac12}, \beta^n = q_t^{n;\frac14} - \overline{\partial}_t q^n .$$

Now define $\hat{\xi}^0 = 0$ and $\hat{\xi}^n = \Delta t\sum_{j=0}^{n-1}\xi^{j+\frac12}$. Then $\partial_t\hat{\xi}^{n+\frac12} = \xi^{n+\frac12}$, $\xi^{n;\frac14} = (\xi^{n+\frac12} + \xi^{n-\frac12})/2$ and $\Delta t\sum_{j=1}^{J}\xi^{j;\frac14} = \hat{\xi}^{J+\frac12} - \frac{\Delta t}{2}\xi^{\frac12}$.

Below, we prove the main theorem of this section.

Theorem 3.1. Assume that U^0 and Z^0 satisfy (21). Then there exists a positive constant $C(\|q\|_{L^\infty(L^\infty)})$ independent of h and Δt such that

$$\max_{0\le J\le N}\| q(t_J) - Z^J\| \le C(\|q\|_{L^\infty(L^\infty)})\Delta t^2(\|q_{ttt}\|_{L^\infty(L^2)} + \|q_{tt}\|_{L^2(L^2)} + \|q_{tttt}\|_{L^2(L^2)})$$
$$+ C(\|q\|_{L^\infty(L^\infty)})h^{\min\{r+1,k\}}(\|u(0)\|_{k+1} + \|u\|_{L^\infty(H^{k+1})} + \|q(0)\|_{r+1} + \|q\|_{L^\infty(H^{r+1})} + \|q\|_{L^2(H^{r+1})}).$$

Proof. Multiplying (22b) by $\Delta t/2$ and noting that $\frac{\Delta t}{2}\partial_t\xi^{\frac12} = \frac{\xi^1+\xi^0}{2} - \xi^0 = \xi^{\frac12} - \xi^0$, we obtain

$$(\partial_t\xi^{\frac12}, w_h) + \kappa(\xi^{\frac12}, w_h) - \kappa(\xi^0, w_h) + A_1(\frac{\Delta t}{2}\xi^{\frac12}, w_h) = -\frac{\Delta t}{2}(\tau^0, w_h) - (\partial_t\rho^{\frac12}, w_h) - \kappa(\frac{\Delta t}{2}\beta^0, w_h)$$
$$-\kappa(\rho^{\frac12} - \rho^0, w_h) + (\frac{\Delta t}{2}\rho^{\frac12} + \frac{\Delta t}{2}\xi^{\frac12}, w_h) - \frac{\Delta t}{2}((q\cos u)^{\frac12} - (Z\cos U)^{\frac12}, w_h). \quad (23)$$

Multiplying (22c) by Δt and summing from n=1 to m, we obtain

$$(\partial_t\xi^{m+\frac12} - \partial_t\xi^{\frac12}, w_h) + \kappa(\xi^{m+\frac12} - \xi^{\frac12}, w_h) + A_1(\hat{\xi}^{m+\frac12} - \frac{\Delta t}{2}\xi^{\frac12}, w_h)$$
$$= \Delta t\sum_{n=1}^{m}(-(\tau^n, w_h) - \kappa(\beta^n, w_h) - ((q\cos u)^{n;\frac14} - (Z\cos U)^{n;\frac14}, w_h)) - (\partial_t\rho^{m+\frac12}, w_h) \quad (24)$$
$$+(\partial_t\rho^{\frac12}, w_h) - \kappa(\rho^{m+\frac12}, w_h) + \kappa(\rho^{\frac12}, w_h) + (\hat{\rho}^{m+\frac12}, w_h) - (\frac{\Delta t}{2}\rho^{\frac12}, w_h) + (\hat{\xi}^{m+\frac12}, w_h) - (\frac{\Delta t}{2}\xi^{\frac12}, w_h).$$

Using (23), we write (24) as

$$(\partial_t\xi^{m+\frac12}, w_h) + \kappa(\xi^{m+\frac12}, w_h) + A_1(\hat{\xi}^{m+\frac12}, w_h) = -(\partial_t\rho^{m+\frac12}, w_h) - \kappa(\rho^{m+\frac12}, w_h) + (\hat{\rho}^{m+\frac12} + \hat{\xi}^{m+\frac12}, w_h)$$
$$+\kappa(\xi^0, w_h) + \Delta t\sum_{n=1}^{m}(-(\tau^n, w_h) - \kappa(\beta^n, w_h) - ((q\cos u)^{n;\frac14} - (Z\cos U)^{n;\frac14}, w_h)) \quad (25)$$
$$+\kappa(\rho^0, w_h) - \frac{\Delta t}{2}(\tau^0, w_h) - \kappa(\frac{\Delta t}{2}\beta^0, w_h) - \frac{\Delta t}{2}((q\cos u)^{\frac12} - (Z\cos U)^{\frac12}, w_h).$$

Denotes $\sum_{n=1}^{0} = 0$, and let $m=0$ in (25), noting that $\hat{\xi}^{\frac12} = \frac{\Delta t}{2}\xi^{\frac12}$, $\hat{\rho}^{\frac12} = \frac{\Delta t}{2}\rho^{\frac12}$, comparing (25) with (23), we obtain that(25) is also tenable even if $m=0$. Further, we note the fact that

$$(\partial_t\xi^{m+\frac12}, \xi^{m+\frac12}) = \frac{1}{2\Delta t}(\|\xi^{m+1}\|^2 - \|\xi^m\|^2), \quad A_1(\hat{\xi}^{m+\frac12}, \partial_t\hat{\xi}^{m+\frac12}) = \frac{1}{2\Delta t}(A_1(\hat{\xi}^{m+1}, \hat{\xi}^{m+1}) - A_1(\hat{\xi}^m, \hat{\xi}^m)).$$

Set $w_h = \xi^{m+\frac12} = \partial_t\hat{\xi}^{m+\frac12}$ in (25), apply the Cauchy-Schwarz's inequality and the Young's inequality to obtain

$$\frac{1}{2\Delta t}(\|\xi^{m+1}\|^2-\|\xi^m\|^2)+\kappa\|\xi^{m+\frac{1}{2}}\|^2+\frac{1}{2\Delta t}(A_1(\hat{\xi}^{m+1},\hat{\xi}^{m+1})-A_1(\hat{\xi}^m,\hat{\xi}^m))$$
$$\le C(\|\xi^0\|^2+\|\rho^0\|^2+\|\rho^{m+\frac{1}{2}}\|^2+\|\hat{\rho}^{m+\frac{1}{2}}\|^2+\|\hat{\xi}^{m+\frac{1}{2}}\|^2+\|\partial_t\rho^{m+\frac{1}{2}}\|^2)+C\Delta t^2\sum_{n=0}^{m}(\|\tau^n\|^2+\|\beta^n\|^2) \quad (26)$$
$$+C\Delta t^2\sum_{n=0}^{m+1}(\|\rho^n\|^2+\|\xi^n\|^2)+C\|q\|^2_{L^\infty(L^\infty)}\Delta t^2\sum_{n=0}^{m+1}(\|\eta^n\|^2+\|\zeta^n\|^2)+\kappa\|\xi^{m+\frac{1}{2}}\|^2.$$

Multiply (26) by $2\Delta t$ and sum from $m=0$ to J with $J+1\le N$ to obtain

$$\|\xi^{J+1}\|^2+A_1(\hat{\xi}^{J+1},\hat{\xi}^{J+1})\le C(\|\xi^0\|^2+\|\rho^0\|^2)+C\Delta t^2\sum_{m=0}^{J}(\|\tau^m\|^2+\|\beta^m\|^2)+C\Delta t^2\sum_{m=0}^{J+1}(\|\rho^m\|^2+\|\xi^m\|^2)$$
$$+C\Delta t\sum_{n=0}^{J}(\|\rho^{m+\frac{1}{2}}\|^2+\|\hat{\rho}^{m+\frac{1}{2}}\|^2+\|\hat{\xi}^{m+\frac{1}{2}}\|^2+\|\partial_t\rho^{m+\frac{1}{2}}\|^2)+C\|q\|^2_{L^\infty(L^\infty)}\Delta t^2\sum_{m=0}^{J+1}(\|\eta^m\|^2+\|\zeta^m\|^2). \quad (27)$$

For the estimate of $\|\zeta^n\|$ in (27), Choose $v_h=4\zeta^{n+1}$ in (22a). Note that $(\nabla\zeta^{n+\frac{1}{2}},4\zeta^{n+1})\ge\|\nabla\zeta^{n+1}\|^2-\|\nabla\zeta^n\|^2$, apply the Cauchy-Schwarz's inequality and the Young's inequality to obtain

$$\|\nabla\zeta^{n+1}\|^2-\|\nabla\zeta^n\|^2\le C(\|\rho^{n+\frac{1}{2}}\|^2+\|\xi^{n+\frac{1}{2}}\|^2)+\frac{1}{2}\|\nabla\zeta^{n+1}\|^2. \quad (28)$$

Summing from $n=0$ to m in (28), we get

$$\|\nabla\zeta^{m+1}\|^2\le\|\nabla\zeta^0\|^2+C\sum_{n=0}^{m}(\|\rho^{n+\frac{1}{2}}\|^2+\|\xi^{n+\frac{1}{2}}\|^2)+\frac{1}{2}\sum_{n=0}^{m-1}\|\zeta^{n+1}\|^2+\frac{1}{2}\|\nabla\zeta^{m+1}\|^2. \quad (29)$$

Noting that $\|\zeta\|_1\le C\|\nabla\zeta\|$ as $\zeta\in H_0^1$ and apply discrete Gronwall's lemma to obtain

$$\|\zeta^{m+1}\|^2\le\|\zeta^{m+1}\|_1^2\le\|\nabla\zeta^{m+1}\|^2\le C(\|\nabla\zeta^0\|^2+\sum_{n=0}^{m}(\|\rho^{n+\frac{1}{2}}\|^2+\|\xi^{n+\frac{1}{2}}\|^2)). \quad (30)$$

Note that

$$\|\phi^{m+\frac{1}{2}}\|^2\le\frac{1}{2}(\|\phi^{m+1}\|^2+\|\phi^m\|^2),\ \ \|\hat{\phi}^{m+\frac{1}{2}}\|^2\le C\Delta t\sum_{j=0}^{m+1}\|\phi^j\|^2.$$

Substituting (30) into (27), using above inequality and the positive definiteness of A_1, we easily conclude that

$$\|\xi^{J+1}\|^2+\|\hat{\xi}^{J+1}\|_1^2\le C(\|\xi^0\|^2+\|\rho^0\|^2+\Delta t\|q\|^2_{L^\infty(L^\infty)}\|\nabla\zeta^0\|^2)+C\Delta t^2\sum_{m=0}^{J}(\|\tau^m\|^2+\|\beta^m\|^2)$$
$$+C\Delta t\sum_{m=0}^{J+1}(\|\rho^m\|^2+\|q\|^2_{L^\infty(L^\infty)}\|\rho^m\|^2+\Delta t\|q\|^2_{L^\infty(L^\infty)}\|\eta^m\|^2+\|\partial_t\rho^{m+\frac{1}{2}}\|^2) \quad (31)$$
$$+C(1+\|q\|^2_{L^\infty(L^\infty)})\Delta t\sum_{m=0}^{J}(\|\xi^m\|^2+\|\hat{\xi}^m\|^2)+C\Delta t\|\hat{\xi}^{J+1}\|^2+C\Delta t\|q\|^2_{L^\infty(L^\infty)}\|\xi^{J+1}\|^2.$$

Now, choose Δt in (31) to satisfy $1-C\Delta t>0$ and $1-C\Delta t\|q\|^2_{L^\infty(L^\infty)}>0$, apply Gronwall's lemma to obtain

$$\|\xi^{J+1}\|^2+\|\hat{\xi}^{J+1}\|_1^2\le C(\|q\|_{L^\infty(L^\infty)})[\|\xi^0\|^2+\|\rho^0\|^2+\|\nabla\zeta^0\|^2$$
$$+\Delta t\sum_{n=0}^{J+1}(\|\rho^m\|^2+\|\eta^m\|^2+\|\partial_t\rho^{m+\frac{1}{2}}\|^2)+\Delta t^2\sum_{m=0}^{J}(\|\tau^m\|^2+\|\beta^m\|^2)]. \quad (32)$$

For the estimates of the right side of (32), we note that

$$\|\tau^0\|^2\le C\Delta t^2\|q_{ttt}\|^2_{L^\infty(L^2)},\|\beta^0\|^2\le C\Delta t^3\int_{t_0}^{t_1}\|q_{ttt}\|^2ds,$$
$$\|\tau^m\|^2\le C\Delta t^3\int_{t_{m-1}}^{t_{m+1}}\|q_{ttt}\|^2ds,\|\beta^m\|^2\le C\Delta t^3\int_{t_{m-1}}^{t_{m+1}}\|q_{ttt}\|^2ds,\|\partial_t\rho^{m+\frac{1}{2}}\|^2\le\frac{C}{\Delta t}\int_{t_m}^{t_{m+1}}\|\rho_t\|^2ds, m\ge1.$$

Further, since $\|\xi^0\|\le\|q(0)-Z^0\|+\|q(0)-\tilde{q}_h(0)\|$ and $\|\zeta^0\|_1\le\|u(0)-U^0\|_1+\|u(0)-\tilde{u}_h(0)\|_1$, it follows that

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$$\| \xi^0 \| \le Ch^{r+1} \| q(0) \|_{r+1}, \ \| \zeta^0 \|_1 \le Ch^k \| u(0) \|_{k+1}.$$

Using above estimates and (10 and (11) in (32), we get

$$\begin{aligned} \| \xi^{J+1} \| \le & C(\| q \|_{L^\infty(L^\infty)}) h^{\min\{r+1,k\}} (\| q(0) \|_{r+1} + \| u(0) \|_{k+1} + \| q \|_{L^\infty(H^{r+1})} + \| u \|_{L^\infty(H^{k+1})} + \| q \|_{L^2(H^{r+1})}) \\ & + C(\| q \|_{L^\infty(L^\infty)}) \Delta t^2 (\| q_{ttt} \|_{L^\infty(L^2)} + \| q_{ttt} \|_{L^2(L^2)} + \| q_{tttt} \|_{L^2(L^2)}). \end{aligned} \quad (33)$$

Finally, apply the triangle inequality with (10) and (33) to complete the proof.

Remark 3.1. Choose $v_h = \zeta^{n+1/2}$ in (22a), apply the Cauchy-Schwarz's inequality and $\| \zeta \|_1 \le C \| \nabla \zeta \|$ as $\zeta \in H_0^1$ to obtain

$$\| \zeta^{n+1/2} \|_1 \le C \| \nabla \zeta^{n+1/2} \| \le C(\| \rho^{n+1/2} \| + \| \xi^{n+1/2} \|).$$

Noting that $\| \xi^{n+1/2} \| \le \frac{1}{2} (\| \xi^{n+1} \| + \| \xi^n \|)$, applying the triangle inequality with (10, (11) and (33), appropriately, it follows that

$$\| u(t_{n+1/2}) - U^{n+1/2} \|_1 + \| q(t_{n+1/2}) - Z^{n+1/2} \| \le C(h^{\min\{r+1,k\}} + \Delta t^2).$$

IV. CONCLUSION

We extend the modified H^1-Galerkin mixed finite element method to solve the damped Sine-Gordon equation. This method does not request LBB-consistency condition and restrict on the approximating spaces. We give the semi-discrete and fully-discrete modified H^1-Galerkin mixed finite element schemes for the model equation in two and three space dimensions, and obtain optimal error estimates for both schemes. When d=2, we obtain an quasi-optimal maximum norm estimates of $u - u_h$.

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