Stability Analysis of Mathematical Syn-Ecological Model Comprising of Prey-Predator, Host-Commensal, Mutualism and Neutral Pairs-IV (One of the Four Species are Washed out States)

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Abstract - This investigation deals with a mathematical model of a four species (S1, S2, S3 and S4) Syn-Ecological system (one of the four species are washed out states). S2 is a predator surviving on the prey S1. The predator S2 is a commensal to the host S3. The pairs S2 and S4, S1 and S3 are neutral. The mathematical model equations characterizing the syn-ecosystem constitute a set of four first order non-linear coupled differential equations. There are in all sixteen equilibrium points. Criteria for the stability of four of the sixteen equilibrium points: One of the four species are washed out states only are established in this paper. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

I. INTRODUCTION
Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmacokinetics are some of those disciplines. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of everyone. Mathematical modeling of ecosystems was initiated by Lotka [9] and by Volterra [18]. The general concept of modeling has been presented in the treatises of Meyer [11], Cushing [4], Paul Colinvaux [11], Freedman [5], Kapur [6, 7]. The ecological interactions can be broadly classified as Prey-Predation, Competition, and Mutualism and so on. N.C. Srinivas [17] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [8] has investigated the two species prey-predator models. Stability analysis of competitive species was carried out by Archana Reddy [3] while Acharyulu [1, 2] investigated Ammensalis between two species. Recently local stability analysis for a two-species ecological mutualism model has been investigated by present author et al [12, 13, 14, 15, 16]. Example for S1, S2, S3 and S4 are Insects, Insectivorous Plants (nephantis, drosera etc.), VAM associated with the plant roots, Soil bacteria respectively.

II. BASIC EQUATIONS
The model equations for a four species multi-system are given by a set of four non-linear ordinary differential equations as

(i) For S1: The Prey of S1 and Neutral to S3
\[ \frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{21}N_1N_2 \]  .... (2.1)

(ii) For S2: The Predator surviving on S1 and Commensal to S3
\[ \frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_2N_1 + a_{23}N_2N_3 \]  .... (2.2)

(iii) For S3: The Host of S2 and Mutual to S4
\[ \frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 + a_{34}N_3N_4 \]  .... (2.3)

(iv) For S4: Mutual to S3 and Neutral to S2
\[
\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^2 + a_{43} N_4 N_3
\]  
\[\text{.... (2.4)}\]

with the following notation.

\(N_i(t)\): Population strengths of the species \(S_i\) at time \(t\), \(i=1, 2, 3, 4\).

\(a_i\): The natural growth rates of \(S_i\), \(i=1, 2, 3, 4\).

\(a_{12}, a_{21}\): Interaction (Prey-Predator) coefficients of \(S_1\) due to \(S_2\) and \(S_2\) due to \(S_1\).

\(a_{34}, a_{43}\): Mutually interaction between \(S_3\) and \(S_4\).

\(K_i, \frac{a_i}{a_{ii}}\): Carrying capacities of \(S_i\), \(i=1, 2, 3, 4\).

Further the variables \(N_1, N_2, N_3, N_4\) are non-negative and the model parameters \(a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24}\) are assumed to be non-negative constants.

### III. EQUILIBRIUM STATES

The system under investigation has sixteen equilibrium states defined by

\[
\frac{dN_i}{dt} = 0, \quad i = 1, 2, 3, 4
\]  
\[\text{...... (3.1)}\]

are given in the following table.

I. Fully washed out state:

\(E_1\): \(N_1 = 0, N_2 = 0, N_3 = 0, N_4 = 0\)

II. States in which three of the four species are washed out and fourth is surviving

\(E_2\): \(N_1 = 0, N_2 = 0, N_3 = 0, N_4 = \frac{a_4}{a_{44}}\)

\(E_3\): \(N_1 = 0, N_2 = 0, N_3 = \frac{a_3}{a_{33}}, N_4 = 0\)

\(E_4\): \(N_1 = 0, N_2 = \frac{a_2}{a_{22}}, N_3 = 0, N_4 = 0\)

\(E_5\): \(N_1 = \frac{a_1}{a_{11}}, N_2 = 0, N_3 = 0, N_4 = 0\)

III. States in which two of the four species are washed out while the other two are surviving

\(E_6\): \(N_1 = 0, N_2 = 0, N_3 = \frac{a_3 a_{34} + a_4 a_{44}}{a_{44} a_{44} - a_{34} a_{34}}, N_4 = \frac{a_3 a_{43} + a_4 a_{33}}{a_{33} a_{44} - a_{34} a_{34}}\)

This state exists only when \(a_{33} a_{44} - a_{34} a_{43} > 0\)

\(E_7\): \(N_1 = 0, N_2 = \frac{a_2}{a_{22}}, N_3 = 0, N_4 = \frac{a_4}{a_{44}}\)

\(E_8\): \(N_1 = 0, N_2 = \frac{a_3}{a_{33}} + \frac{a_2}{a_{33}}, N_3 = \frac{a_3}{a_{33}}, N_4 = 0\)

\(E_9\): \(N_1 = \frac{a_3}{a_{11}}, N_2 = 0, N_3 = 0, N_4 = \frac{a_4}{a_{44}}\)

\(E_{10}\): \(N_1 = \frac{a_3}{a_{11}}, N_2 = 0, N_3 = \frac{a_3}{a_{33}}, N_4 = 0\)
IV. States in which one of the four species is washed out while the other three are surviving

\[ \overline{N}_1 = 0, \overline{N}_2 = \frac{a_2(a_4a_{34} + a_3a_{44})}{a_{22}(a_3a_{44} - a_{34}a_{43})} + \frac{a_2}{a_{22}}, \overline{N}_3 = 0, \overline{N}_4 = \frac{a_4a_{34} + a_3a_{44}}{a_{33}a_{44} - a_{34}a_{43}}. \]

This state exists only when \( a_3a_{44} - a_{34}a_{43} > 0 \)

IV. The co-existent state (or) Normal steady state

\[ \overline{N}_1 = \frac{\beta_1}{\beta_2}, \overline{N}_2 = \frac{\beta_3}{\beta_4}, \overline{N}_3 = \frac{a_3}{a_{33}}, \overline{N}_4 = 0 \]

Where

\[ \beta_1 = a_{33}(a_4a_{22} + a_1a_{21}), \beta_4 = a_{33}(a_4a_{22} - a_2a_{12}) - a_4a_{22}, \beta_3 = a_{33}(a_4a_{22} - a_2a_{12}) + a_4a_{22}, \beta_4 = a_{33}(a_4a_{22} + a_1a_{21}) \]

This state exists only when \( \beta_2 > 0 \)

V. The present paper deals with one of the four species are washed out states only. The stability of the other equilibrium states will be presented in the forth coming communications.

IV. STABILITY OF ONE OF THE FOUR SPECIES WASHED OUT EQUILIBRIUM STATES:

V. (SL. NOS 12,13,14,15 IN THE ABOVE EQUILIBRIUM STATES)

A. Stability of the Equilibrium State \( E_{12} \)

Let us consider small deviations \( u_1(t), u_2(t), u_3(t), u_4(t) \) from the steady state i.e.

\[ \overline{N}_i(t) = \overline{N}_i + u_i(t), i = 1,2,3,4 \] --- (4.1.1)
Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of \( u_1, u_2, u_3, u_4 \), we get
\[
\frac{du_1}{dt} = w_1 u_1 \quad \text{... (4.1.2)}
\]
\[
\frac{du_2}{dt} = a_{21} N_3 u_1 - w_2 u_2 + a_{23} N_1 u_3 \quad \text{... (4.1.3)}
\]
\[
\frac{du_3}{dt} = a_{31} N_3 u_1 - a_{34} N_4 u_4 \quad \text{... (4.1.4)}
\]
\[
\frac{du_4}{dt} = a_{43} N_4 u_3 - a_{44} N_4 u_4 \quad \text{... (4.1.5)}
\]

Here \( w_1 = a_1 - a_{12} N_2 \), \( w_3 = a_2 + a_{23} N_3 \) \( \text{... (4.1.6)} \)

The characteristic equation of which is
\[
(\lambda - w_1)(\lambda + w_3) \left[ \lambda^2 + (a_{33} N_3 + a_{44} N_4)\lambda + (a_{33} a_{44} - a_{34} a_{43}) N_3 N_4 \right] = 0 \quad \text{... (4.1.7)}
\]

The characteristic roots of (4.1.6) are
\[
\lambda = w_1, \lambda = -w_3, \lambda = \frac{- (a_{33} N_3 + a_{44} N_4) \pm \sqrt{(a_{33} N_3 - a_{44} N_4)^2 + 4 a_{34} a_{43} N_3 N_4}}{2}
\]

**Case (A):** If \( w_1 < 0 \) [i.e. \( a_1 < a_{12} N_2 \)]

Here \( w_2, w_3 \) are negative and the other two roots are also negative.

Hence the equilibrium state is stable.

The solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are
\[
u_1 = u_{10} e^{\nu t} \quad \text{... (4.1.8)}
\]
\[
u_2 = \left[ u_{20} - \frac{a_{23} N_1 u_{10}}{(\lambda_3 - \lambda_4)} \right] e^{\nu t} + \frac{a_{23} N_4 (P_1 + P_2)}{(\lambda_3 - \lambda_4)} + \frac{a_{23} N_3 P_2 e^{\nu t} + a_{23} N_3 P_2 e^{\nu t}}{(\lambda_3 - \lambda_4)}
\]
\[
u_3 = \left[ u_{30} \left( \lambda_3 + a_{44} N_4 \right) + u_{40} a_{34} N_3 \right] e^{\nu t} + \frac{u_{30} \left( \lambda_4 + a_{33} N_3 \right) + u_{40} a_{34} N_3}{\lambda_4 - \lambda_3} e^{\nu t} \quad \text{... (4.1.9)}
\]
\[
u_4 = \left[ u_{40} \left( \lambda_3 + a_{44} N_3 \right) + u_{40} a_{43} N_3 \right] e^{\nu t} + \frac{u_{40} \left( \lambda_4 + a_{33} N_3 \right) + u_{40} a_{43} N_3}{\lambda_4 - \lambda_3} e^{\nu t} \quad \text{... (4.1.10)}
\]

Where \( P_1 = \frac{u_{30} \left( \lambda_3 + a_{44} N_4 \right) + u_{40} a_{34} N_3}{\lambda_3 - \lambda_4} \), \( P_2 = \frac{u_{40} \left( \lambda_4 + a_{44} N_4 \right) + u_{40} a_{43} N_3}{\lambda_4 - \lambda_3} \)

Where \( u_{10}, u_{20}, u_{30}, u_{40} \) are the initial values of \( u_1, u_2, u_3, u_4 \) respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates \( a_1, a_2, a_3, a_4 \) and the initial values of the perturbations \( u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t) \) of the species \( S_1, S_2, S_3, S_4 \). Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

The solutions are illustrated in figures.

**Case (i):** If \( u_{30} < u_{20} < u_{10} < u_{40} \) and \( a_3 < a_1 < a_2 < a_4 \)

In this case initially \( S_1 \) dominates the Prey \( (S_1) \) and the Predator \( (S_3) \) till the time instant \( t_1, t_2, t_3, t_4 \) respectively and thereafter the dominance is reversed. And \( u_1, u_2, u_3, u_4 \) are converging asymptotically to the equilibrium point. Hence the equilibrium point is stable.
Case (ii): If \( u_{40} < u_{10} < u_{30} < u_{20} \) and \( A_2 < a_3 < a_1 < a_4 \)

In this case initially the Predator (S_2) dominates the Prey (S_i) till the time instant \( t^*_{12} \) and thereafter the dominance is reversed. Also the host (S_3) of S_2 dominates the Prey (S_i) and S_4 till the time instant \( t^*_{13}, t^*_{43} \) respectively and the dominance gets reversed thereafter.

![Fig. 2](image)

Case (B): If \( w_i > 0 \) [i.e. \( a_i > a_{12}N_2 \)]

Here the root \( w_i \) is positive and the other three roots are negative.

Hence the equilibrium state is **unstable** and the solutions in this case are same as in Case (A).

The solutions are illustrated in figures.

Case (i): If \( u_{30} < u_{40} < u_{20} < u_{10} \) and \( a_1 < a_4 < A_2 < a_3 \)

In this case initially S_4 dominates the host (S_3) of S_2, the Prey (S_i) and the Predator (S_2) in natural growth rate as well as in its initial population strength.

![Fig. 3](image)

Case (ii): If \( u_{10} < u_{30} < u_{20} < u_{40} \) and \( a_1 < a_3 < a_4 < A_2 \)

In this case initially S_4 dominates the Predator (S_2) and the Prey (S_i) till the time instant \( t^*_{24}, t^*_{14} \) respectively and thereafter the dominance is reversed. Also the host (S_3) of S_2 dominates the Prey (S_i) till the time instant \( t^*_{13} \) and the dominance gets reversed thereafter. And the Predator (S_2) dominates the Prey (S_i) till the time instant \( t^*_{13} \) and the dominance gets reversed thereafter.
B. Stability of the Equilibrium State $E_{13}$:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of $u_1$, $u_2$, $u_3$, $u_4$, we get

\[ \frac{du_1}{dt} = -a_1 u_1 - a_{12} \overline{N}_4 u_2 \] ... (4.2.1)
\[ \frac{du_2}{dt} = g_2 u_2 \] ... (4.2.2)
\[ \frac{du_3}{dt} = -a_{33} \overline{N}_3 u_3 + a_{34} \overline{N}_4 u_4 \] ... (4.2.3)
\[ \frac{du_4}{dt} = a_{43} \overline{N}_4 u_3 - a_{44} \overline{N}_4 u_4 \] ... (4.2.4)

Here $g_2 = a_3 + a_{23} \overline{N}_1 + a_{23} \overline{N}_3$ ... (4.2.5)

The characteristic equation of which is

\[ (\lambda + a_1)(\lambda - g_2)\left[ \lambda^2 + (a_{33} \overline{N}_3 + a_{44} \overline{N}_4)\lambda + (a_{33}a_{44} - a_{34}a_{43})\overline{N}_3 \overline{N}_4 \right] = 0 \] ... (4.2.6)

The characteristic roots of (4.2.6) are

\[ \lambda = -a_1, \lambda = g_2, \lambda = \frac{-(a_{33} \overline{N}_3 + a_{44} \overline{N}_4) \pm \sqrt{(a_{33} \overline{N}_3 - a_{44} \overline{N}_4)^2 + 4a_{34}a_{43} \overline{N}_3 \overline{N}_4}}{2} \]

One root of the characteristic equation $\lambda = r_2$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

\[ u_1 = \left( u_{10} + \frac{a_{12} a_4 u_{20}}{a_{11} (g_2 + a_1)} \right) e^{-a_1 t} - \frac{a_{12} a_4 u_{20}}{a_{11} (g_2 + a_1)} e^{r_2 t} \] ... (4.6.7)
\[ u_2 = u_{20} e^{r_2 t} \] ... (4.6.8)
\[ u_3 = p_1 e^{\lambda_3 t} + p_2 e^{\lambda_4 t} \] ... (4.6.9)
\[ u_4 = \left[ \frac{u_{40} (\lambda_3 + a_{33} \overline{N}_3) + u_{30} a_{43} \overline{N}_4}{\lambda_3 - \lambda_4} \right] e^{\lambda_3 t} + \left[ \frac{u_{40} (\lambda_4 + a_{34} \overline{N}_3) + u_{30} a_{43} \overline{N}_4}{\lambda_4 - \lambda_3} \right] e^{\lambda_4 t} \] ... (4.6.10)

Where

\[ p_1 = \frac{u_{30} (\lambda_3 + a_{43} \overline{N}_3) + u_{40} a_{43} \overline{N}_3}{\lambda_3 - \lambda_4}, p_2 = \frac{u_{30} (\lambda_4 + a_{44} \overline{N}_4) + u_{40} a_{44} \overline{N}_4}{\lambda_4 - \lambda_3} \]

The solutions are illustrated in figures.

**Case (i):** If $u_{30} < u_{30} < u_{10} < u_{30}$ and $a_4 < a_3 < a_1 < g_2$

In this case initially the host ($S_3$) of $S_2$ dominates the Prey ($S_1$) till the time instant $t_{23}$ and the dominance gets reversed thereafter. Also the Prey ($S_1$) dominates the Predator ($S_2$) till the time instant $t_{21}$ and thereafter the dominance is reversed.

![Fig. 5](image-url)
Case (ii): If $u_{10} < u_{20} < u_{40} < u_{30}$ and $g_2 < a_1 < a_4 < a_3$

In this case initially the host ($S_3$) of $S_2$ dominates the Predator ($S_2$) till the time instant $t_{23}$ and thereafter the dominance is reversed. Also $S_4$ dominates the Predator ($S_2$) till the time instant $t_{24}$ and the dominance gets reversed thereafter.

**C. Stability of the Equilibrium State $E_{14}$**

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of $u_1, u_2, u_3, u_4$, we get

$$\frac{du_1}{dt} = -a_{11} \overline{N_1} u_1 - a_{12} \overline{N_1} u_2$$

$$\frac{du_2}{dt} = (a_2 - 2a_{22} \overline{N_2} + a_{21} \overline{N_1}) u_2 + a_{21} \overline{N_2} u_1 + a_{23} \overline{N_2} u_3$$

$$\frac{du_3}{dt} = l_3 u_3$$

$$\frac{du_4}{dt} = -a_4 u_4 + a_{43} \frac{a_4}{a_{44}} u_4$$

Here $l_3 = a_3 + a_{34} \frac{a_4}{a_{44}}$

The characteristic equation of which is

$$(\lambda - l_3)(\lambda + a_3)[\lambda^2 - (M_3 - a_{11} \overline{N_1}) \lambda + \left[(-a_{11} \overline{N_1})(M_3) + a_{12} a_{21} \overline{N_1} \overline{N_2}\right] = 0$$

The characteristic roots of (4.3.6) are

$$\lambda = l_3, \lambda = -a_4,$$

$$\lambda = \frac{(M_3 - a_{11} \overline{N_1}) \pm \sqrt{(M_3 - a_{11} \overline{N_1})^2 - [a_{12} a_{21} \overline{N_1} \overline{N_2} - M_3 a_{11} \overline{N_1}]}}{2}$$

Where $M_3 = a_2 - 2a_{22} \overline{N_2} + a_{21} \overline{N_1}$

One root of the characteristic equation $\lambda = l_3$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

$$u_1 = \left\{ \left( u_{10} + u_{20} \right) a_{12} \overline{N_1} - \phi \left( \lambda_2 - l_3 \right) \right\} e^{\lambda_3 t}$$

$$+ \left\{ \left( u_{10} - \phi \right) \lambda_2 - \phi \right\} e^{\lambda_3 t} + \phi e^{\lambda_3 t}$$
\[ u_2 = \left( \frac{u_{10} + u_{20}}{\lambda_2 - \lambda_1} \right) a_{12} N_1 - \phi_3 (\lambda_3 - l_3) e^{\lambda t} \xi_1 \]
\[ + \left( \frac{u_{10} - \phi_3 (\lambda_2 - \lambda_1) - (u_{10} + u_{20}) a_{12} N_1 + \phi_3 (\lambda_2 - l_2)}{\lambda_2 - \lambda_1} \right) e^{\lambda t} \xi_2 + \phi_4 e^{\lambda t} \]

\[ u_3 = u_{30} e^{l_3 t} \]

\[ u_4 = \left( \frac{a_4 a_{43} u_{30}}{a_{44} (l_3 + a_4)} \right) e^{a_4 t} + \frac{a_4 a_{43} u_{30}}{a_{44} (l_3 + a_4)} e^{l_4 t} \]

\[ \phi_3 = \frac{\beta_4}{l_3 + \psi_1 l_3 + \beta_1} \]
\[ \psi_2 = a_{11} N_1 - a_{21} N_1 + 2 a_{22} N_2 - a_{22} \overline{N_2} = a_{12} \overline{N_1} \]
\[ \phi_4 = \frac{\phi_4 (l_3 + p_3)}{a_{41} N_1} \]
\[ \beta_3 = (2 a_{11} a_{22} N_2 - a_{22} a_{11} - a_{12} a_{21} N_1 - a_{12} a_{21} \overline{N_1}) N_1, \beta_4 = -a_{12} a_{23} u_{30} N_1 \overline{N_2} \]
\[ \xi_3 = \frac{-(\lambda_3 + p_3)}{a_{12} N_1}, \xi_2 = \frac{-(\lambda_2 + p_3)}{a_{12} N_1}, p_3 = a_{11} \overline{N_1} \]

The solutions are illustrated in figures.

**Case (i):** If \( u_{30} < u_{20} < u_{40} < u_{10} \) and \( l_3 < a_4 < a_2 < a_4 \)

In this case initially the Prey (\( S_1 \)) dominates \( S_4 \) and the host (\( S_3 \)) of \( S_2 \) till the time instant \( t_{41}, t_{31} \) respectively and thereafter the dominance is reversed. Also \( S_4 \) dominates the host (\( S_3 \)) of \( S_2 \) till the time instant \( t_{34} \) and the dominance gets reversed thereafter. Similarly the Predator (\( S_2 \)) dominates the host (\( S_3 \)) of \( S_2 \) till the time instant \( t_{32} \) and thereafter the dominance is reversed.

**Case (ii):** If \( u_{10} < u_{30} < u_{40} < u_{20} \) and \( a_4 < a_1 < a_2 < a_4 \)

In this case initially the Predator (\( S_2 \)) dominates \( S_4 \), the host (\( S_3 \)) of \( S_2 \) and the Prey (\( S_1 \)) of \( S_2 \) till the time instant \( t_{42}, t_{32}, t_{12} \) respectively and thereafter the dominance is reversed. Also \( S_4 \) dominates the host (\( S_3 \)) of \( S_2 \) till the time instant \( t_{34} \) and the dominance gets reversed thereafter.
D. Stability of the Equilibrium State $E_{15}$:

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of $u_1, u_2, u_3, u_4$, we get

$$\frac{du_1}{dt} = -a_{11} N_1 u_1 - a_{12} \overline{N_1} u_2 \quad \ldots \quad (4.4.1)$$

$$\frac{du_2}{dt} = -a_{22} N_2 u_2 + a_{21} \overline{N_1} u_1 + a_{23} \overline{N_2} u_3 \quad \ldots \quad (4.4.2)$$

$$\frac{du_3}{dt} = -a_4 u_3 + a_{43} \overline{N_3} u_4 \quad \ldots \quad (4.4.3)$$

$$\frac{du_4}{dt} = n_4 u_4 \quad \ldots \quad (4.4.4)$$

Here $n_4 = a_4 + a_{43} \frac{a_1}{a_{33}} \ldots \quad (4.4.5)$

The characteristic equation of which is

$$[\lambda^2 + (a_{11} N_1 + a_{22} \overline{N_2})\lambda + (a_{12} a_{23} + a_{14} a_{43}) \overline{N_1 N_2}](\lambda - n_4)(\lambda + a_4) = 0 \quad \ldots \quad (4.4.6)$$

The characteristic roots of (4.15.6) are

$$\lambda = \frac{- (a_{11} N_1 + a_{22} \overline{N_2}) \pm \sqrt{(a_{11} N_1 - a_{22} \overline{N_2})^2 - 4 a_{12} a_{23} \overline{N_1 N_2}}}{2}, \lambda = n_4, \lambda = -a_4$$

One root of the characteristic equation $\lambda = n_4$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

$$u_1 = \left\{ \frac{(u_{10} + u_{20} - w_3^* - w_4^*) a_{12} \overline{N_1} - (\lambda_2 + P_3)(w_3 + w_4)}{\lambda_2 - \lambda_4} \right\} e^{\lambda_4 t}$$

$$+ \left\{ \frac{(u_{10} - (w_3 + w_4)) (\lambda_2 - \lambda_4) + (u_{10} + u_{20} - w_3^* - w_4^*) a_{12} \overline{N_1} - (\lambda_2 + P_3)(w_3 + w_4)}{\lambda_2 - \lambda_4} \right\} e^{\lambda_4 t} + w_3^* e^{-a_4 t} + w_4^* e^{a_4 t} \quad \ldots \quad (4.4.7)$$

$$u_2 = \left\{ \frac{(u_{30} + u_{40} - w_3^* - w_4^*) a_{12} \overline{N_1} - (\lambda_2 + P_3)(w_3 + w_4)}{\lambda_2 - \lambda_4} \right\} e^{\lambda_4 t}$$

$$+ \left\{ \frac{(u_{30} - (w_3 + w_4)) (\lambda_2 - \lambda_4) + (u_{30} + u_{40} - w_3^* - w_4^*) a_{12} \overline{N_1} - (\lambda_2 + P_3)(w_3 + w_4)}{\lambda_2 - \lambda_4} \right\} e^{\lambda_4 t} + w_3^* e^{-a_4 t} + w_4^* e^{a_4 t} \quad \ldots \quad (4.4.8)$$

$$u_3 = \left[ u_{30} - \frac{a_3 a_{43} u_{40}}{a_{33} (n_4 + a_3)} \right] e^{-a_4 t} + \frac{a_3 a_{43} u_{40}}{a_{33} (n_4 + a_3)} e^{a_4 t} \quad \ldots \quad (4.4.9)$$

$$u_4 = u_{40} e^{a_4 t} \quad \ldots \quad (4.4.10)$$

where

$$w_3 = \frac{(u_{30} - r_3 a_{23} \overline{N_1 N_2})}{D^2 - \psi_1 D + \beta_3}, w_4 = \frac{a_{12} a_{23} \overline{N_1 N_2}}{D^2 - \psi_1 D + \beta_3}, D = -a_4$$

$$w_3^* = w_3 (a_3 - P_3), w_4^* = -(n_4 + P_3), \xi_1 = \frac{- (\lambda_1 + P_3)}{a_{12} \overline{N_1}}, \xi_2 = \frac{- (\lambda_2 + P_3)}{a_{12} \overline{N_1}}, P_3 = a_{11} \overline{N_1}$$
\[ \psi_i = a_{i1}N_i + a_{i2}N_2, \quad \beta_i = a_{i1}N_i + a_{i2}N_2, \quad r_i = \frac{a_i a_{i3} u_{40}}{a_{i3}(n_4 + a_i)} \]

The solutions are illustrated in figures.

**Case (i):** If \( u_{40} < u_{40} < u_{10} < u_{20} \) and \( n_4 < a_2 < a_2 < n_1 \)

In this case initially the Predator (\( S_2 \)) dominates \( S_4 \) and the host (\( S_3 \)) of \( S_2 \) till the time instant \( t^*_{24} \), \( t^*_{32} \) respectively and thereafter the dominance is reversed. Also the Prey (\( S_1 \)) dominates \( S_4 \) and the host (\( S_3 \)) of \( S_2 \) till the time instant \( t^*_{41}, t^*_{31} \) respectively and the dominance gets reversed thereafter.

![Fig. 9](image)

**Case (ii):** If \( u_{40} < u_{40} < u_{10} < u_{20} \) and \( n_4 < a_3 < a_2 < n_1 \)

In this case initially the host (\( S_3 \)) of \( S_2 \) dominates \( S_4 \) till the time instant \( t^*_{43} \) and thereafter the dominance is reversed. Also the Predator (\( S_2 \)) dominates \( S_4 \) till the time instant \( t^*_{24} \) and the dominance gets reversed thereafter. Similarly the Prey (\( S_1 \)) dominates \( S_4 \) till the time instant \( t^*_{41} \) and thereafter the dominance is reversed.

![Fig. 10](image)

**REFERENCES**


